Wind Racer Redux

Revisiting a Probability Experiment Game

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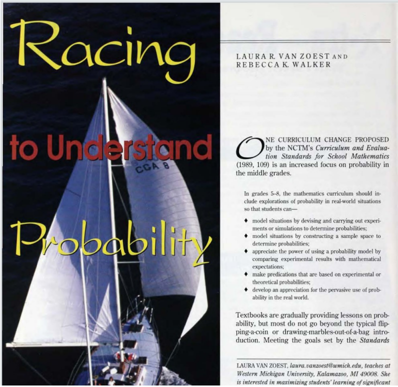
Introducing the Wind Racer game

Dr. Sue Haller from St Cloud State University

Introducing the Wind Racer game

The original article can be found here:

Van Zoest, L. R., & Walker, R. K. (1997). *Racing to understand probability.* Mathematics Teaching in the Middle School, 3(2), 162-170.



LAURA R. VAN ZOEST AND
REBECCA K. WALKER

ONE CURRICULUM CHANGE PROPOSED by the NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989, 109) is an increased focus on probability in the middle grades.

> In grades 5–8, the mathematics curriculum should include explorations of probability in real-world situations so that students can—
>
> - model situations by devising and carrying out experiments or simulations to determine probabilities;
> - model situations by constructing a sample space to determine probabilities;
> - appreciate the power of using a probability model by comparing experimental results with mathematical expectations;
> - make predications that are based on experimental or theoretical probabilities;
> - develop an appreciation for the pervasive use of probability in the real world.

Textbooks are gradually providing lessons on probability, but most do not go beyond the typical flipping-a-coin or drawing-marbles-out-of-a-bag introduction. Meeting the goals set by the *Standards*

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Conjecturing

I'm going to pass out the materials for the game and run through how we are going to play it.

Have a look at the materials for the game and then I want you to conjecture in your groups about what will happen when we play this game. Record your conjectures by putting your initials next to the sailboat you think is going to win!

Playing the Wind Racer Game

Please scan this QR code or visit

`tinyurl.com/WindRacerData`

to view our data!

We're playing Game 1 right now, so you'll want to view under that tab.

Observations

1. What did we notice in that game?
2. Would we consider it *fair*? And how do we understand *fairness*?
3. How do we make it fair *keeping the same boats*?

Introducing *Proportional* Wind Racer

The original article makes the point that if we want to keep the "boats" the same, students – with some nudging – eventually come to the idea that we make the tracks different length, adjusting the length to be proportional to their probabilities (or to make the moves different lengths, which serves the same purpose of making the game more proportional).

The idea is that the boats which have a greater chance of being rolled should have further to go and the boats with a smaller chance of being rolled have shorter to go, making the game more fair... and hopefully completely fair!

Conjecturing

So let's try this again:

Have a look at the materials for the game and then I want you to conjecture in your groups about what will happen when we play this game. Record your conjectures by putting your initials next to the sailboat you think is going to win with this new game board!

Playing the Wind Racer Game

Please scan this QR code or visit

`bit.ly/WindRacerData`

to observe our data!

Use the tab for "Game 2" for this game, please.

Observations

1. What did we notice in that game?
2. Does it seem fair? Why or why not?
3. What can we do to further adjust this game to better understand how fair it is/isn't?

The Original Ending...

In the original article, the conclusion is made that the game is now fair. This is supported by a classroom experiemnt with a very small n value.

In Dr Haller's demonstration, we saw that it didn't seem to be fair, but nobody really understood why. It was also based on a very small n value, but the data seemed compelling.

And for all of the years that I have done this in my secondary Math Methods course at Augsburg, we see that it doesn't seem to be fair, and just sort of left it at that.

… and where my work began…

I made the decision to go down that rabbit-hole and investigate this problem.

… but where to begin?

The problem seemed really daunting and a ton of cases, so I honestly had very little idea where to start.

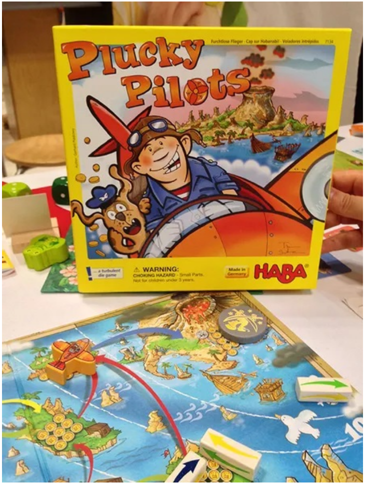
- Guess and check
- Make an orderly list
- Eliminate possibilities
- Use symmetry
- Consider special cases
- Use direct reasoning
- Solve an equation
- Look for a pattern
- Draw a picture
- Solve a simpler problem
- Use a model
- Work backwards
- Use a formula
- Be ingenious

Introduce *d4 Proportional* Wind Racer

I decided to go with the approach of *solving a simpler problem*.

I did this by changing from 6-sided dice (d6) to 4-sided dice (d4). This would reduce the number of outcomes as well as making the game shorter.

Side Note on Games with 4 Sided Dice



Conjecturing

So let's try this again:

Have a look at the materials for the game and then I want you to conjecture in your groups about what will happen when we play this game. Record your conjectures by putting your initials next to the sailboat you think is going to win with this new game board!

Playing the *d4 Proportional* Wind Racer Game

Please scan this QR code or visit

`bit.ly/WindRacerData`

to observe our data!

Use the tab for “Game 3” for this game, please.

Observations

1. What did we notice in that game?
2. Is it more or less fair than the d6 version? How do we even measure that or make that comparison?
3. And how can we use this game to inform how we understand the other games?

First, the Monte Carlo Simulations

The first approach was to write Python code to play the 'proportional' and fixed length games millions of times. I wrote the code for the d4, d6, d8 and d20 games. This is what I found after 10 million completed proportional d4 games:

Outcome	2 or 8	3 or 7	4 or 6	5
Observed Probability	35.2428%	23.2586%	18.2347%	23.2638%

A one-way Chi-square test on these results against the fair, uniform distribution showed a *p*-value of effectively zero.

Brute Force Attack on the d4 Game Probabilities

Let's consider the d4 game where 2/8 wins.

The shortest game is two rolls, where we just roll 2/8 both times and the game ends; the longest possible game is 13 rolls, three 3/7s, five 4/6s and three fives in addition to the two 2s... but most importantly, a 2/8 has to be rolled last!

If I code the 2/8 roll as an a, the 3/7 roll as a b, the 4/6 roll as a c and the 5 roll as a d, I can start coding these outcomes as monomials. The two roll win could be written as a^2 and the longest possible game would be $a^2b^3c^5d^3$.

Brute Force Attack on the d4 Game Probabilities

If each monomial represents a case, then there must be some polynomial which represents the probability.
My first instinct was to take the sum

$$\sum_{i=2}^{13} (a+b+c+d)^i$$

but not every term is going to appear, like a^{13}, which is an impossible outcome since the $2/8$ can't be rolled 13 times. It's also a lot of terms to calculate, without any real clear picture or systematic way of knowing which terms to remove.

But while this approach wasn't going to work, it did help me to clarify the structure of the problem and break it into three cases:

Proportional d4 Brute Force Attack: Case 1

The first case would be where only one outcome is being rolled. Since the *a* outcome has to win, that's the only outcome that we need to consider.

That two roll game can only happen one possible way, so we know the probability for that case $(2, 0, 0, 0)$ would be

$$1 \cdot (\frac{2}{16})^2 \cdot (\frac{4}{16})^0 \cdot (\frac{6}{16})^0 \cdot (\frac{4}{16})^0 = .015625$$

.

This is the only outcome for Case 1, but it helps us to see the structure of the later cases, so its importance is huge.

Proportional d4 Brute Force Attack: Case 2

The second case is where we have the *a* outcome and only one other outcome.

For example, let's consider the a^2b^1 game. It could happen only two different ways: (a, b, a) *or* (b, a, a). [Take a second to consider why (a, a, b) is not a possible case.] So we can write a^2b^1 as $2a^2b^1$, to account for the number of cases.

Proportional d4 Brute Force Attack: Case 2

More generally, we have that for a Case 2 game with k rolls which a wins, we have the binomial

$$\binom{k-1}{1} = \frac{k-1!}{1!(k-2)!} = (k-1)$$

as the number of cases for that game. The one is because there is only 1 free position for the first a outcome since the second outcome is fixed as being last.

For example, the game with 5 rolls of c and 2 rolls of a could happen $\binom{6}{1} = 6$ ways, giving us the term $6a^2c^5$ and the probability

$$6 \cdot (\frac{2}{16})^2 \cdot (\frac{4}{16})^0 \cdot (\frac{6}{16})^5 \cdot (\frac{4}{16})^0 \approx .000695229$$

Proportional d4 Brute Force Attack: Case 2

I want to point out that the binomial seems trivial now, but gets slightly more complicated for the other outcomes.

For example, we are not considering this case right now, but for games with k rolls where the b outcome wins, we would have the binomial

$$\binom{k-1}{3} = \frac{k-1!}{3!(k-4)!}$$

since b needs four rolls to win and with the last roll fixed as the winning roll, we have three rolls which can appear anywhere else in the outcomes.

Proportional d4 Brute Force Attack: Case 3

The third and final case is where we have the *a* outcome and only two or more other outcomes.

For example, let's consider the $a^2b^1c^1d^2$ game. Once we start to list these out, we find that we have more outcomes to consider, but also the risk of overcounting the d outcomes. So we have 5 possible outcomes for the first roll, then four, then three, and so on... but we have to divide out the two d outcomes repeating, so we have $\binom{5!}{2} = 60$ outcomes for this game, giving the probability

$$60 \cdot (\frac{2}{16})^2 \cdot (\frac{4}{16})^1 \cdot (\frac{6}{16})^1 \cdot (\frac{4}{16})^2 \approx 0.00549316$$

Proportional d4 Brute Force Attack: Case 3

As other outcomes begin to repeat, we have to build our understanding of how to count these outcomes.

For example, if we look at the game with the outcome $a^2b^3c^5d^2$, we need to remove the fixed a outcome, and then we have 11 outcomes for the first roll, down to one. We also have three b outcomes, which can appear $b!$ ways, so we need to divide not by 3, but by 3!. We also need to divide by 5! for the c outcomes and 2! for the d outcomes. This gives us $\frac{11!}{3!5!2!} = 27720$ outcomes for this game and the probability

$$27720 \cdot (\frac{2}{16})^2 \cdot (\frac{4}{16})^3 \cdot (\frac{6}{16})^5 \cdot (\frac{4}{16})^2 \approx 0.00313668$$

Proportional d4 Brute Force Attack: Case 3

Some of you may have noticed that this coefficient is known as a multinomial and is the coefficient when we do an exponential expansion of a polynomial. It is calculated just like the binomial we saw in Case 2, but accounts for more than just two outcomes.

More generally, we have that for a Case 3 game with k rolls which a wins, we have the multinomial

$$\binom{k-1}{1\ x\ y\ z} = \frac{k-1!}{1!x!y!z!}$$

as the number of cases for that game.

Brute Force Attack on the d4 Game Probabilities

Once all of that was in place, I was able to start listing out cases and calculating probabilities on Excel. This also helped me to understand how to structure the sub-case progression, which would be essential in writing the code, which was the next step.

Having these probabilities also helped me to check the code along the way.

	A	B	C	D	E	F	G
1							
2		2 or 8	3 or 7	4 or 6	7		
3		0.125	0.25	0.375	0.25		
4	max rolls	2	3	5	3	cases	probability
5	2 roll game	2	0	0	0	1	0.015625
6	3 roll game	2	0	0	1	2	0.0078125
7		2	0	1	0	2	0.01171875
8		2	1	0	0	2	0.0078125
9	4 roll game	2	0	0	2	3	0.00292969
10		2	0	1	1	6	0.00878906
11		2	0	2	0	3	0.0065918
12		2	1	0	1	6	0.00585938
13		2	1	1	0	6	0.00878906
14		2	2	0	0	3	0.00292969
15	5 roll game	2	0	0	3	4	0.00097656
16		2	0	1	2	12	0.00439453
17		2	0	2	1	12	0.0065918
18		2	0	3	0	4	0.0032959
19		2	1	0	2	12	0.00292969
20		2	1	1	1	24	0.00878906
21		2	1	2	0	12	0.0065918
22		2	2	0	1	12	0.00292969
23		2	2	1	0	12	0.00439453
24		2	3	0	0	4	0.00097656

Extrapolation into a d4 Calculation Code

The final step was to use what I understood about the structure of the problem to then write code to generate all of the possible cases and calculate the actual probability.

Outcome	2 or 8	3 or 7	4 or 6	5
Obs Probability	35.2428%	23.2586%	18.2347%	23.2638%
Exp Probability	35.2181%	23.2719%	18.2381%	23.2719%

As you might imagine, the one-way Chi-square *p*-value comparing observed v expected data was effectively 1. We can now see conclusively that this game is **NOT** fair.

Further Extrapolation into a d6 Calculation Code

Once I understood how to construct the d4 code, I was able to write the d6 code. It was slightly more complicated as there were a few more cases to consider on the iterations, but I eventually got these results:

Outcome	2 or 12	3 or 11	4 or 10	5 or 9	6 or 8	7
Obs Prob	32.27%	19.54%	14.08%	10.99%	9.04%	14.08%
Exp Prob	32.27%	19.55%	14.07%	11.00%	9.04%	14.07%

Again, the one-way Chi-square *p*-value was effectively 1 between observed and experimental data. We can see conclusively that this game is also **NOT** fair.

Results and Conclusions

- Switching the problem from d6 to d4 was key. It reduced the total number of cases from approximately $2.35x10^{19}$ for the d6 game to 'only' $1,041,808$ in the d4 game.
- Making the tracks proportionally longer makes the game *more* fair, but the game remains imbalanced.
- For games using dice with more faces, the same observations hold: the shorter the winning track, the more rolls are possible and so the greater the winning percentage! The low probability is overcompensated by more trials. For example, in the d20 case, using $n = 1,000,000$, 2 and 40, which can only be rolled two ways won 30.1695% of the games, while 20 and 22 only won 0.5794% of the games, despite being able to be rolled 38 different ways!

Results and Conclusions

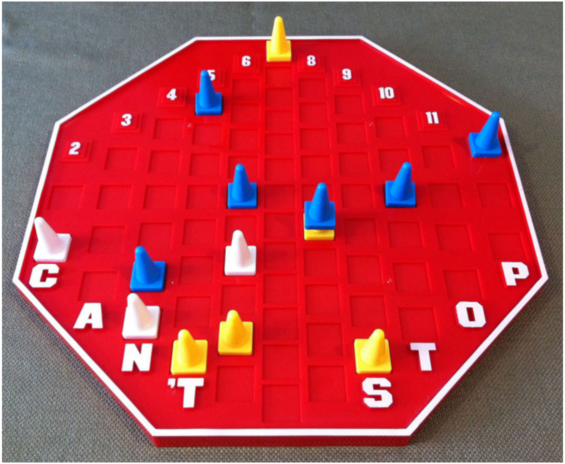
- Another important takeaway was a reminder of the importance of the *fundamentals of problem solving*. This wasn't the most difficult problem I have ever solved, but it was maybe one of the most complex because of so many cases to consider and understand. Good problem solving practice scales easily.
- Maybe the most important result was that *published results can be wrong*. It's certainly not to say that every published result is wrong, but if you feel like there's something that can be added or changed to a published result, then pursue it.

Implications for Further Research

- So how *do* we make the game fair by adjusting the game board? Is there a game board structure which a chi-square test will show as 'fair' at $\alpha = 0.05$? What's the smallest board which will accomplish that for d4?
- What are the theoretical probabilities for the other dice (d8, d10, d12 and d20)? And what does that mean for the smallest possible 'fair' game board for those dice?
- There has to be another way to calculate this by hand rather than using computers. I suspect there is a generating function approach that works that I want to investigate.

Implications for Further Research

- Does this result imply an optimal strategy for Sid Sackson's game *Can't Stop*, which uses a very similar structure?



Questions or Comments?

Please feel free to reach me at

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or visit my webpage at

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for copies of the programs used for this and all of my data and results!