# The State of Pure Shear*

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**Abstract**

In classical continuum mechanics a state of pure shear is defined as one for which there is some orthonormal basis relative to which the normal components of the Cauchy stress tensor vanish. An equivalent characterization is that the trace of the Cauchy stress tensor must vanish. We give an elementary but complete discussion of this fundamental theorem here from both the geometric and algebraic point of view.

KEYWORDS: continuum mechanics, pure shear.

## Pure Shear.

In classical continuum mechanics the Cauchy stress tensor $\mathbf{T}$ is symmetric, and a state of stress is said to be one of pure shear if there is an orthonormal basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ for which

$$T_{aa} \equiv \mathbf{a} \cdot (\mathbf{Ta}) = 0 \ \ , \ T_{bb} \equiv \mathbf{b} \cdot (\mathbf{Tb}) = 0 \ \ , \ T_{cc} \equiv \mathbf{c} \cdot (\mathbf{Tc}) = 0 \ \ . \tag{1}$$

It is well-known[†]that a necessary and sufficient condition for $\mathbf{T}$ to be a state of pure shear is

$$\mathrm{tr}\mathbf{T} = 0 \ \ . \tag{2}$$

While it suffices for the proof of this result to exhibit just one orthonormal basis for which (2) $\Rightarrow$ (1), our purpose here is to give an elementary argument which exhibits all such bases. We prove the following theorem.

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†Love [2], §16, mentioned this as a property of the strain quadric in the linear theory of elasticity for strains with "cubical dilatation" but did not prove it. In his classic book on tensor analysis, McConnel [3], Ch.6, problem 10, noted this result in an exercise for traceless symmetric tensors. More recently, Gurtin [1], pp. 36-37, provided a proof that (2) $\Rightarrow$ (1) within the context of infinitesimal strain theory, which he attributes to J. Lew. Gurtin does not characterize the complete set of orthonormal bases for which (1) follows from (2), but rather shows the existence of but one, which suffices to prove the result. In 1960, the great teacher and elastician Professor Eli Sternberg brought this theorem to the attention of his graduate class during his course of lectures on linear elasticity at Brown University.

**Theorem 1** *Let* $\mathbf{T} \in \mathrm{Sym} \subset \mathbb{R}^3 \otimes \mathbb{R}^3$ *have the spectral form* $\mathbf{T} = \sigma_1 \mathbf{n}_1 \otimes \mathbf{n}_1 + \sigma_2 \mathbf{n}_2 \otimes \mathbf{n}_2 + \sigma_3 \mathbf{n}_3 \otimes \mathbf{n}_3$. *Let* $\mathrm{tr}\mathbf{T} = 0$, *and suppose*[‡] *that*

$$\sigma_1 > 0 \ , \ \sigma_2 \geq 0 \ , \ \sigma_3 < 0 \ . \tag{3}$$

*Let* $\mathbf{x} = x_i \mathbf{n}_i \in \mathbb{R}^3$, *and consider the elliptical cone*[§]

$$\mathbb{C} \ : \ \ \mathbf{x} \cdot (\mathbf{T}\mathbf{x}) = \sigma_1 x_1^2 + \sigma_2 x_2^2 - |\sigma_3| x_3^2 = 0 \tag{4}$$

*of Figure 1. Then, corresponding to each unit vector* $\mathbf{a}$ *which lies along a generator of the cone from its tip, there is an orthonormal basis* $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ *for which* (1) *holds. If* $\det\mathbf{T} \neq 0$ *(i.e.,* $\sigma_2 \neq 0$*), this basis is unique up to the replacement* $\mathbf{a} \to \pm\mathbf{a}, \mathbf{b} \to \pm\mathbf{b}, \mathbf{c} \to \pm\mathbf{c}$. *If* $\det\mathbf{T} = 0$ *(i.e.,* $\sigma_2 = 0$*), the unique set of bases for which* (1) *holds is given by* $\mathbf{a} = \pm(\mathbf{n}_1 \pm \mathbf{n}_3)/\sqrt{2}$ *and any two orthonormal vectors,* $\mathbf{b}$ *and* $\mathbf{c}$*, which lie in the plane whose normal is* $\mathbf{a}$.

**Proof** (Geometric). An important geometrical property of the elliptical cone $\mathbb{C}$ is that its opening angle cannot be less that $90°$. To see this, it suffices to observe that because of (4) the elliptical cone $\mathbb{C}$ at height $x_3$ cuts off an ellipse whose minimum diameter is

$$2\sqrt{\frac{|\sigma_3|}{\sigma_1}} x_3 = 2\sqrt{\frac{\sigma_1 + \sigma_2}{\sigma_1}} x_3 \geq 2x_3 \ \ ,$$

where "=" applies only if $\sigma_2 = 0$. Thus, the cone has a minimum opening angle of $90°$. It is equal to $90°$ only if $\sigma_2 = 0$ in which case the cone degenerates into the union of two orthogonal planes $\mathbb{P}_1 \cup \mathbb{P}_2$ as noted in the footnote above and illustrated in Figure 2. Thus, in the case $\sigma_2 \neq 0$, if we let $\mathbf{a}$ be any vector along a generator of the cone drawn through the origin (see Figure 3) then plane $\mathbb{P}$ orthogonal to $\mathbf{a}$ will intersect the cone uniquely along two lines which also are generators of the cone. These two generators define two corresponding unit vectors $\mathbf{b}$ and $\mathbf{c}$ as shown in Figure 3, both of which are orthogonal to $\mathbf{a}$. Of course, these vectors are unique up to the replacement $\mathbf{b} \to \pm\mathbf{b}$, $\mathbf{c} \to \pm\mathbf{c}$. It is clear that (1) holds for the vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ since they all lie on the surface of the cone $\mathbb{C}$. But, because of $(1)_{1,2}$, and because $\mathrm{tr}\mathbf{T} = 0$ and $\{\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}\}$ is an orthonormal basis for $\mathbb{R}^3$, we then know that

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{T}(\mathbf{a} \times \mathbf{b}) = 0 \ \ .$$

Thus $\mathbf{a} \times \mathbf{b} \in \mathbb{P} \cup \mathbb{C}$, and because the plane $\mathbb{P}$ intersects the cone $\mathbb{C}$ *uniquely* along two lines containing $\mathbf{b}$ and $\mathbf{c}$, respectively, then

$$\mathbf{c} = \pm\mathbf{a} \times \mathbf{b}$$

and so $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\} \subset \mathbb{C}$ is an *orthonormal* basis for which (1) holds. This basis is unique up to the replacement $\mathbf{a} \to \pm\mathbf{a}$, $\mathbf{b} \to \pm\mathbf{b}$, $\mathbf{c} \to \pm\mathbf{c}$, and the claim of the theorem is justified in the case that $\det\mathbf{T} \neq 0$.

In the case $\sigma_2 = 0$ the cone $\mathbb{C}$ degenerates into the union of two perpendicular planes $\mathbb{P}_1 \cup \mathbb{P}_2$ parallel to the eigenvector $\mathbf{n}_2$ and containing the vectors $(\mathbf{n}_1 \pm \mathbf{n}_3)/\sqrt{2}$, as shown

[‡]If $\mathrm{tr}\mathbf{T} = 0$, then either $\mathbf{T}$ or $-\mathbf{T}$ will satisfy (1) so the condition (3) is without loss of generality.

[§]If $\sigma_2 = 0$ then $\mathrm{tr}\mathbf{T} = 0 \Rightarrow \sigma_3 = -\sigma_1$ and the cone (4) degenerates into the union of two orthogonal planes $\mathbb{P}_1 \cup \mathbb{P}_2$ which contain the $\mathbf{n}_2$-axis and which are at $45°$ between the $\mathbf{n}_1$- and $\pm\mathbf{n}_3$-directions, respectively (see Figure 2).

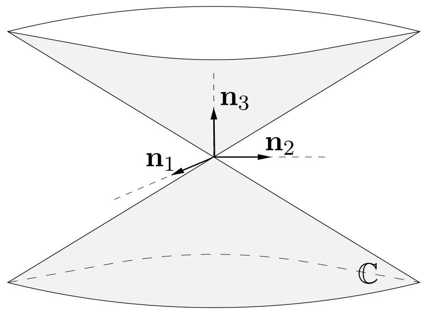


Figure 1: The cone $\mathbb{C}$ and the principal axes of $\mathbf{T}$.

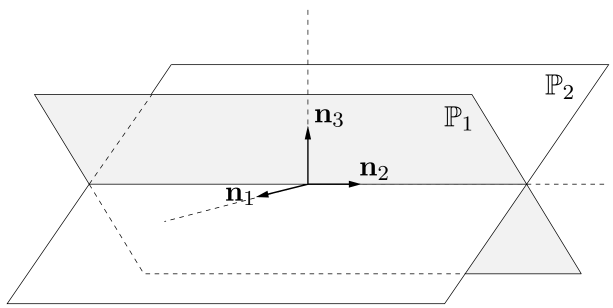


Figure 2: The degenerate cone of orthogonal planes $\mathbb{P}_1 \cup \mathbb{P}_2$ for $\sigma_2 = 0$.

in Figure 2. If $\mathbf{a}$ is any unit vector in $\mathbb{P}_1$ other than $\pm(\mathbf{n}_1 + \mathbf{n}_3)/\sqrt{2}$, it is clear that a plane $\mathbb{P}$ orthogonal to $\mathbf{a}$ will intersect $\mathbb{P}_1 \cup \mathbb{P}_2$ uniquely in two orthogonal lines; one containing the unit vectors $\pm(\mathbf{n}_1 - \mathbf{n}_3)/\sqrt{2} \in \mathbb{P}_2$ and the other containing the unit vectors $\pm\mathbf{a} \times (\mathbf{n}_1 - \mathbf{n}_3)/\sqrt{2} \in \mathbb{P}_1$, as shown in Figure 4. Thus, defining $\mathbf{b} \equiv (-\mathbf{n}_1 + \mathbf{n}_3)/\sqrt{2}$ and $\mathbf{c} \equiv \mathbf{a} \times (-\mathbf{n}_1 + \mathbf{n}_3)/\sqrt{2}$, we see that $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\} \subset \mathbb{P}_1 \cup \mathbb{P}_2$ is an orthonormal basis for which (1) holds. This basis is unique up to the replacement $\mathbf{a} \to \pm\mathbf{a}$, $\mathbf{b} \to \pm\mathbf{b}$, $\mathbf{c} \to \pm\mathbf{c}$.

On the other hand, if $\mathbf{a} = \pm(\mathbf{n}_1 + \mathbf{n}_3)/\sqrt{2} \in \mathbb{P}_1$ then *any* two orthogonal unit vectors in $\mathbb{P}_2$, say $\mathbf{b}$ and $\mathbf{c} \equiv \mathbf{a} \times \mathbf{b}$, will define an orthonormal basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\} \subset \mathbb{P}_1 \cup \mathbb{P}_2$ for which (1) holds. This basis is unique up to the replacement $\mathbf{a} \to \pm\mathbf{a}$, $\mathbf{b} \to \pm\mathbf{b}$, $\mathbf{c} \to \pm\mathbf{c}$.

Analogous to the above conclusions for the case $\mathbf{a} \in \mathbb{P}_1$, we can give similar arguments for the case $\mathbf{a} \in \mathbb{P}_2$. Thus, by a simple re-ordering of the vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ we justify the claim of the theorem in the case that $\det\mathbf{T} = 0$. ♣

**Proof** (Algebraic). We shall use the following well-known formulae from linear algebra for any $\mathbf{T} \in \mathrm{Sym}$ which satisfies $\mathrm{tr}\mathbf{T} = 0$:

$$\left.\begin{aligned} (\mathrm{Cof}\mathbf{T})\mathbf{T} = \mathbf{T}(\mathrm{Cof}\mathbf{T}) &= \mathbf{1}\det\mathbf{T} \ , \\ \mathrm{Cof}\mathbf{T} &= \mathrm{II}\mathbf{1} + \mathbf{T}^2 \ , \\ \mathbf{T}(\mathbf{u} \times \mathbf{T}\mathbf{u}) &= ((\mathrm{Cof}\mathbf{T})\mathbf{u}) \times \mathbf{u} \quad \forall \mathbf{u} \in \mathbb{R}^3 \ , \end{aligned}\right\} \tag{5}$$

where II is the second principal invariant of $\mathbf{T}$.

Now, except for those unit vectors $\mathbf{a}$ that satisfy $(1)_1$ and are such that $\mathbf{T}\mathbf{a} = \mathbf{0}$,[¶] we

[¶]If $\det\mathbf{T} \neq 0$ there are no such unit vectors. If $\det\mathbf{T} = 0$ then according to (3) $\sigma_2 = 0$ and then

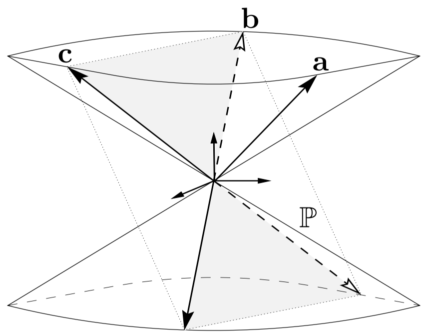


Figure 3: The cone $\mathbb{C}$ and intersecting plane $\mathbb{P} \perp \mathbf{a}$.

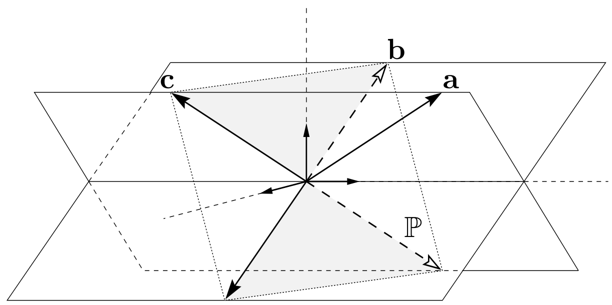


Figure 4: The degenerate cone of orthogonal planes $\mathbb{P}_1 \cup \mathbb{P}_2$ and $\mathbb{P} \perp \mathbf{a}$.

introduce the orthogonal basis $\{\mathbf{a}, \mathbf{Ta}, \mathbf{a} \times \mathbf{Ta}\}$ and set

$$\mathbf{e} = \alpha\mathbf{Ta} + \beta\mathbf{a} \times \mathbf{Ta} \ , \tag{6}$$

where $\alpha$ and $\beta$ are scalars yet to be defined. It is clear that $\mathbf{e} \perp \mathbf{a}$ and our aim is to determine all $\alpha$ and $\beta$ such that

$$\mathbf{e} \cdot \mathbf{Te} = 0 \ , \ |\mathbf{e}| = 1 \ . \tag{7}$$

First, note that with the aid of (5), (6) and $(1)_1$ we have

$$\mathbf{Te} = \alpha\mathbf{T}^2\mathbf{a} + \beta(\mathbf{T}^2\mathbf{a}) \times \mathbf{a} \ .$$

Then, trivially,

$$\mathbf{e} \cdot \mathbf{Te} = \mathbf{T}^2\mathbf{a} \cdot (\alpha\mathbf{e} + \beta\mathbf{a} \times \mathbf{e}) \ .$$

But, because of $(1)_1$, (6) and the fact that $|\mathbf{a}| = 1$, it follows that

$$\mathbf{a} \times \mathbf{e} = \alpha\mathbf{a} \times \mathbf{Ta} - \beta\mathbf{Ta} \ ,$$

and, again with the aid of (6), we may write

$$\mathbf{e} \cdot \mathbf{Te} = \alpha\mathbf{T}^2\mathbf{a} \cdot (\alpha\mathbf{Ta} + \beta\mathbf{a} \times \mathbf{Ta}) + \alpha\beta\mathbf{T}^2\mathbf{a} \cdot (\mathbf{a} \times \mathbf{Ta}) - \beta^2\mathbf{T}^2\mathbf{a} \cdot \mathbf{Ta} \ .$$

$\mathrm{tr}\mathbf{T} = 0 \Rightarrow \sigma_3 = -\sigma_1$. In this case $\mathbf{T} = \sigma_1(\mathbf{n}_1 \otimes \mathbf{n}_1 - \mathbf{n}_3 \otimes \mathbf{n}_3)$ and the null manifold of $\mathbf{T}$ corresponds to the line parallel to $\mathbf{n}_2$. Thus, $\mathbf{Ta} = \mathbf{0}$ only for those unit vectors $\mathbf{a} = \pm\mathbf{n}_2$. This special case arises only if $\det\mathbf{T} = 0$ and we shall consider it, subsequently.

Because $\mathbf{T} \in \mathrm{Sym}$, this readily simplifies to

$$\mathbf{e} \cdot \mathbf{T}\mathbf{e} = \alpha^2 \mathbf{a} \cdot \mathbf{T}^3 \mathbf{a} + 2\alpha\beta(\mathbf{a} \times \mathbf{T}\mathbf{a}) \cdot \mathbf{T}^2 \mathbf{a} - \beta^2 \mathbf{a} \cdot \mathbf{T}^3 \mathbf{a} \ .$$

Now, because of $(5)_{1,2}$, $(1)_1$ and the fact that $|\mathbf{a}| = 1$ we readily see that

$$\mathbf{a} \cdot \mathbf{T}^3 \mathbf{a} = \det \mathbf{T} \ ,$$

and so it follows that

$$\mathbf{e} \cdot \mathbf{T}\mathbf{e} = \alpha^2 \det \mathbf{T} + 2\alpha\beta\gamma - \beta^2 \det \mathbf{T} \ , \tag{8}$$

where

$$\gamma \equiv (\mathbf{a} \times \mathbf{T}\mathbf{a}) \cdot \mathbf{T}^2 \mathbf{a} \ . \tag{9}$$

Consider the case $\det \mathbf{T} \neq 0$ (i.e., $\sigma_2 \neq 0$): Here, $(7)_1$ and (8) require

$$\alpha = \beta\alpha_\pm \ , \ \alpha_\pm \equiv \frac{-\gamma \pm \sqrt{\gamma^2 + (\det \mathbf{T})^2}}{\det \mathbf{T}} \ ,$$

and we see that there are only two solutions of $(7)_1$ which have the form (6); they are given by

$$\mathbf{e}_\pm = \beta(\alpha_\pm \mathbf{T}\mathbf{a} + \mathbf{a} \times \mathbf{T}\mathbf{a}) \ .$$

Moreover, using $(1)_1$ we readily find that

$$\mathbf{e}_- \cdot \mathbf{e}_+ = 0 \ .$$

Therefore, the unique set of unit vectors $\mathbf{b}$ and $\mathbf{c}$ which satisfy (6) is given by

$$\mathbf{b} \equiv \pm \frac{\pm\alpha_+ \mathbf{T}\mathbf{a} + \mathbf{a} \times \mathbf{T}\mathbf{a}}{|\mathbf{T}\mathbf{a}|\sqrt{\alpha_+^2 + 1}} \ , \ \mathbf{c} \equiv \pm \frac{\pm\alpha_- \mathbf{T}\mathbf{a} + \mathbf{a} \times \mathbf{T}\mathbf{a}}{|\mathbf{T}\mathbf{a}|\sqrt{\alpha_-^2 + 1}} \ . \tag{10}$$

Clearly, $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is an orthonormal basis for which (1) holds and it is unique up to the replacement $\mathbf{a} \to \pm\mathbf{a}, \mathbf{b} \to \pm\mathbf{b}, \mathbf{c} \to \pm\mathbf{c}$, and this completes the proof in the case that $\det \mathbf{T} \neq 0$.

Consider the case $\det \mathbf{T} = 0$ (i.e., $\sigma_2 = 0$): Here, we have $\sigma_3 = -\sigma_1, \mathbf{T} = \sigma_1(\mathbf{n}_1 \otimes \mathbf{n}_1 - \mathbf{n}_3 \otimes \mathbf{n}_3)$, and those unit vectors that satisfy $(1)_1$ are then given by

$$\mathbf{a} = a_1(\mathbf{n}_1 \pm \mathbf{n}_3) + a_2 \mathbf{n}_2 \ , \ 2a_1^2 + a_2^2 = 1 \ . \tag{11}$$

Thus, we see that

$$\left.\begin{aligned} \mathbf{T}\mathbf{a} &= \sigma_1 a_1(\mathbf{n}_1 \pm \mathbf{n}_3) \ , \\ \mathbf{a} \times \mathbf{T}\mathbf{a} &= \sigma_1 a_1(\pm 2a_1 \mathbf{n}_2 - a_2 \mathbf{n}_3 \mp a_2 \mathbf{n}_1) \ , \end{aligned}\right\} \tag{12}$$

and using (9) we find that

$$\gamma = \mp 2a_1^2 a_2 \sigma_1^3 \ . \tag{13}$$

Consequently, if $a_1 \neq 0$ then $\mathbf{a}$ is not in the null space of $\mathbf{T}$ and the above analysis applies. Thus, from (8) and (13) we must have

$$\alpha\beta a_2 = 0 \ . \tag{14}$$

If $a_2 = 0$, then $\alpha$ and $\beta$ are arbitrary. In this case, we see from (11) that

$$\mathbf{a} = \pm\frac{\mathbf{n}_1 \pm \mathbf{n}_3}{\sqrt{2}} \quad , \tag{15}$$

and we readily find from (6), (7) and (12) that

$$\mathbf{e} = e_1(\mathbf{n}_1 \pm \mathbf{n}_3) \pm e_2\mathbf{n}_2 \ \ , \ 2e_1^2 + e_2^2 = 1 \quad . \tag{16}$$

Thus, (15) and any two orthonormal vectors, $\mathbf{b}$ and $\mathbf{c}$, in the plane orthogonal to $\mathbf{a}$ will satisfy (1) and the theorem holds.

If $a_2 \neq 0$ (but $a_1 \neq 0$ still holds), then (14) requires either $\alpha = 0$ or $\beta = 0$. In this case $\mathbf{a}$ is of the form (11) and, corresponding to the respective cases $\alpha = 0$ and $\beta = 0$, we see from (6), (7) and (12) that

$$\mathbf{e} = \pm\frac{a_2(\mathbf{n}_1 \pm \mathbf{n}_3) - 2a_1\mathbf{n}_2}{\sqrt{2a_2^2 + 4a_1^2}} \equiv \mathbf{b} \quad , \tag{17}$$

and

$$\mathbf{e} = \pm\frac{\mathbf{n}_1 \mp \mathbf{n}_3}{\sqrt{2}} \equiv \mathbf{c} \quad . \tag{18}$$

Thus, by a simple re-ordering of $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ as defined in (11), (17) and (18), we can recover the same solution as expressed by (15) and (16).

Finally, in the case $\det\mathbf{T} = 0$, we suppose that $a_1 = 0$. In this case, we see from (11) that

$$\mathbf{a} = \pm\mathbf{n}_2 \quad , \tag{19}$$

and, from (12), that $\mathbf{a}$ is in the null space of $\mathbf{T}$. Thus, the main analysis of this theorem does not apply. But, analogous to (6), we may represent any vector orthogonal to $\mathbf{a}$ in the form

$$\mathbf{e} = \alpha\mathbf{n}_1 + \beta\mathbf{n}_3 \quad .$$

Then, we wish to find $\alpha$ and $\beta$ such that (7) holds. Clearly,

$$\mathbf{e} \cdot \mathbf{Te} = \sigma_1(\alpha^2 - \beta^2) \quad ,$$

and (7) is satisfied only in the cases

$$\mathbf{e} = \pm\frac{\mathbf{n}_1 \pm \mathbf{n}_3}{\sqrt{2}} \quad . \tag{20}$$

By re-ordering the orthonormal basis defined in (19) and (20), we again recover special cases covered by the solution of (15) and (16). The proof is complete. ♣

# References


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[2] Love, A.E.H., *A Treatise on the Mathematical Theory of Elasticity,* $4^{th}$ *ed.*. Cambridge University Press, 1927.

[3] McConnel, *Tensor Analysis.*