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THE Γ-CONVERGENCE OF A SHARP INTERFACE THIN FILM MODEL WITH NON-CONVEX ELASTIC ENERGY

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ABSTRACT. We give results for the Γ-limit of a scaled elastic energy of a film as the thickness $h > 0$ converges to zero. The elastic energy density models materials with multiple phases or variants and is thus non-convex. The model includes an interfacial energy that allows sharp interfaces between the phases and variants and is proportional to the total variation of the deformation gradient.

1. INTRODUCTION

Thin films of martensitic crystals are the subject of increasing scientific and technological interest [6,16,22]. Dimensionally reduced models that replace the three-dimensional bulk energy with a two-dimensional thin film energy can make the design of applications more tractable and the computation of the deformation more efficient. New challenges arise in the derivation of thin film energies for martensitic crystals since the presence of multiple phases and variants requires that the elastic energy density be non-convex [4,25,29] and since an interfacial energy that allows sharp interfaces is often useful for accurate modeling [10–12]. Related work on the general problem of rigorously deriving dimensionally reduced energy functionals has been given in [1,3,6,17,19,27,30].

We present results for the Γ-limit [7, 26] of the scaled elastic energy of a thin film with deformation $\tilde{u} : \Omega_h \to \mathbb{R}^3$ defined on a reference domain of thickness $h > 0$ given by $\Omega_h = S \times (-h/2,\, h/2)$ for $S \subset \mathbb{R}^2$ and subject to boundary conditions

$$\tilde{u}(x_1, x_2, x_3) = y_0(x_1, x_2) + b_0(x_1, x_2)x_3 \quad \text{for } (x_1, x_2, x_3) \in \gamma \times (-h/2,\, h/2) \tag{1.1}$$

so the film adheres on a part of its lateral boundary, $\gamma \times (-h/2,\, h/2) \subset \partial S \times (-h/2,\, h/2)$. The elastic energy of the film is given by

$$\mathcal{E}_h(\tilde{u}) = \kappa \int_{\Omega_h} |D(\nabla \tilde{u})| + \int_{\Omega_h} \phi(\nabla \tilde{u}(x), x)\, dx, \tag{1.2}$$

where the term $\kappa \int_{\Omega_h} |D(\nabla \tilde{u})|$ for $\kappa > 0$ models the interfacial energy between phases and variants (the total variation of the deformation gradient is precisely defined in Section 2), and the term $\int_{\Omega_h} \phi(\nabla \tilde{u}(x), x)\, dx$ models the elastic energy of the film. Since we are interested in modeling and computing the deformation of thin films that undergo structural phase transformation, the energy density, $\phi(F, x)$, is generally a non-convex function of the deformation gradient, $F \in \mathbb{R}^{3\times 3}$. The explicit dependence of the energy density, $\phi(F, x)$, on $x \in \Omega_h$ allows the modeling of alloys with compositional fluctuation [11–13, 16, 21].

We rescale the deformations $\tilde{u} : \Omega_h \to \mathbb{R}^3$ to deformations on a fixed domain of thickness one, $u : \Omega_1 \to \mathbb{R}^3$ by

$$u(z_1, z_2, z_3) = \tilde{u}(z_1, z_2, hz_3) \quad \text{for } z = (z_1, z_2, z_3) \in \Omega_1,$$

and we determine and analyze the Γ-limit of the rescaled energy

$$\mathcal{E}_1^{(h)}(u) = \frac{1}{h}\, \mathcal{E}_h(\tilde{u})$$

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subject to rescaled boundary conditions

$$u(z_1, z_2, z_3) = y_0(z_1, z_2) + b_0(z_1, z_2)hz_3 \quad \text{for } (z_1, z_2, z_3) \in \gamma \times (-1/2,\, 1/2). \tag{1.3}$$

We analyze the Γ-limit of $\mathcal{E}_1^{(h)}(u)$ with respect to two related definitions of convergence for deformations. For the first definition, we prove that the Γ-limit of $\mathcal{E}_1^{(h)}(u)$ is given by

$$\mathcal{E}^{(0)}(y, b) = \kappa \left[\int_S |D(\nabla y|\sqrt{2}\, b)| + \sqrt{2} \int_\gamma |b - b_0| \right] + \int_S \phi(\nabla y(\hat{z})|b(\hat{z}), \hat{z}, 0)\, d\hat{z} \tag{1.4}$$

for $y : S \to \mathbb{R}^3$ such that $y = y_0$ on $\gamma \subset \partial S$ and $b : S \to \mathbb{R}^3$. The matrix-valued function $(\nabla y|b) : S \to \mathbb{R}^{3\times 3}$ in the thin film limit (1.4) models the thin film deformation gradient. We also identify $(\hat{z}, 0) \in \mathbb{R}^3$ with $\hat{z} \in S$. For the second definition, we prove that the Γ-limit of $\mathcal{E}_1^{(h)}(u)$ is given by

$$\mathcal{E}_1^{(0)}(u) = \begin{cases} \min_b \mathcal{E}^{(0)}(u_M, b) & \text{if } u_{,3} = 0 \text{ a.e. in } \Omega_1, \\ +\infty & \text{otherwise,} \end{cases}$$

where u_M is the deformation of the midplane, $u_M(z_1, z_2) = u(z_1, z_2, 0)$. For both definitions of convergence of deformations, we give compactness results and show that the uniform coerciveness of the energy functionals $\mathcal{E}_1^{(h)}(u)$ allows us to prove that subsequences of energy-minimizing deformations of $\mathcal{E}_1^{(h)}(u)$ converge to minimizers of the Γ-limit as $h \to 0$.

We have used the thin film energy (1.4) to compute the quasi-static evolution of a martensitic thin film subject to a varying temperature field [8, 9]. In these computations, we use continuation methods for which the film need only be in a local minimum. We think that the results in this paper, especially the Γ-convergence described in Theorem 5.1, justify the use of the thin film energy (1.4) in this context because the result (5.10) guarantees that any admissible (y, b) defined on S can be used to construct an admissible $\tilde{u}_h$ defined on Ω_h that is a "smoothed" version of the deformation

$$y(x_1, x_2) + b(x_1, x_2)x_3 \quad \text{for } (x_1, x_2, x_3) \in \Omega_h,$$

such that $\mathcal{E}^{(0)}(y, b)$ is approximated by $\frac{1}{h}\mathcal{E}_h(\tilde{u}_h)$.

The energy density, $\phi(F, x)$, in models for crystals which undergo a structural phase transformation is not quasi-convex [4, 5, 23–25, 28, 29]. The Γ-limit with respect to weak $W^{1,p}$ convergence of a scaled elastic energy that does not include interfacial energy will generally thus involve the quasi-convexification of the elastic energy density [7, 17]. However, the interfacial energy $\kappa \int_{\Omega_h} |D(\nabla \tilde{u})|$ in our model allows us to obtain sequences of deformations with gradients that converge strongly and to use the strong continuity of the scaled elastic energy $h^{-1} \int_{\Omega_h} \phi(\nabla \tilde{u}(x), x)\, dx$. A related result has been obtained in [6] for a diffuse interfacial energy $\kappa \int_{\Omega_h} |\nabla^2 \tilde{u}|^2\, dx$.

We have recently computed the hysteresis of a martensitic thin film by the application of a thermal and loading cycle [13] using the Γ-limit of the energy

$$\begin{aligned} \hat{\mathcal{E}}_h(\tilde{u}) &= \kappa \int_{\Omega_h} |D(\nabla \tilde{u})| + \int_{\Omega_h} \phi(\nabla \tilde{u}(x), x)\, dx - \int_{\partial\Omega_h} (Tn) \cdot \tilde{u} \\ &= \kappa \int_{\Omega_h} |D(\nabla \tilde{u})| + \int_{\Omega_h} \phi(\nabla \tilde{u}(x), x)\, dx - \int_{\Omega_h} T \cdot \nabla \tilde{u}, \end{aligned} \tag{1.5}$$

where the dead load is Tn for a constant $T \in \mathbb{R}^{3\times 3}$ at points on the boundary $\partial\Omega_h$ with unit exterior normal vector n. The Γ-limit is shown in this paper to be

$$\hat{\mathcal{E}}_1^{(0)}(u) = \begin{cases} \min_b \hat{\mathcal{E}}^{(0)}(u_M, b) & \text{if } u_{,3} = 0 \text{ a.e. in } \Omega_1, \\ +\infty & \text{otherwise,} \end{cases}$$

with

$$\hat{\mathcal{E}}^{(0)}(y, b) = \kappa \int_S |D(\nabla y|\sqrt{2}\, b)| + \int_S \phi(\nabla y(\hat{z})|b(\hat{z}), \hat{z}, 0)\, d\hat{z} - \int_S T \cdot (\nabla y|b).$$

In Section 2, we recall the total variation of functions of bounded variation and give a few needed properties. In Section 3, we describe the assumed properties of the elastic energy density, and in Section 4 we recall the definition of Γ-convergence. The main results and analysis for the Γ-limit of the film model with Dirichlet boundary conditions are given in Section 5, and the main results and analysis for the Γ-limit of the film model with loading boundary conditions are given in Section 6.

Our results in this paper extend the analysis given in [10] by proving the Γ-convergence of the scaled energy functional for the adhering boundary condition (1.1). We also extend the class of energy densities, $\phi(F, x)$, to allow compositional variation, and we extend the class of boundary conditions that can be analyzed by giving results for the Γ-limit of the scaled energy functional with dead loads (1.5).

2. Functions of Bounded Variation

We will assume that $S \subset \mathbb{R}^2$ is a bounded domain with a Lipschitz continuous boundary, ∂S, and denote the reference undistorted configuration of the thin film of the martensitic material by Ω_h, $0 < h \leq 1$, where

$$\Omega_h = S \times (-h/2,\, h/2).$$

The deformations of the thin film are given by functions $\tilde{u} : \Omega_h \to \mathbb{R}^3$ with gradient $\nabla\tilde{u} : \Omega_h \to \mathbb{R}^{3\times3}$. We use the notation $\tilde{u}_{i,j} = \partial\tilde{u}_i/\partial x_j$, and we denote the columns of $\nabla\tilde{u}$ by $\tilde{u}_{,i}$, $i = 1, 2, 3$. The "planar" gradient of $\tilde{u}$, denoted by $\nabla_P\tilde{u} : \Omega_h \to \mathbb{R}^{3\times2}$, has columns given by $\tilde{u}_{,1}$ and $\tilde{u}_{,2}$.

Given an open set $\Omega \subset \mathbb{R}^3$ and a function $v \in L^1(\Omega; \mathbb{R})$, we define the total variation of v [18, 20] by

$$\int_\Omega |Dv| = \sup\left\{\int_\Omega \sum_{k=1,2,3} v(x)\psi_{k,k}(x)\,dx :\ \psi \in \mathcal{C}_0^\infty(\Omega; \mathbb{R}^3),\ |\psi(x)| \leq 1 \ \text{ for all } x \in \Omega\right\}$$

and say $v \in BV(\Omega)$ if $\int_\Omega |Dv| < +\infty$. We recall that $\mathcal{C}_0^\infty(\Omega; \mathbb{R}^3)$ denotes the space of infinitely differentiable functions compactly supported in Ω, whose range is $\mathbb{R}^3$, and we note that $|\psi(x)|$ denotes the usual euclidian norm, that is, the square root of the sum of the squares of all the components of $\psi(x)$.

For a matrix-valued function $v \in L^1(\Omega; \mathbb{R}^{m\times p})$, we define

$$\int_\Omega |Dv| = \sup\left\{\sum_{\substack{i=1,\dots,m\\ j=1,\dots,p\\ k=1,2,3}} \int_\Omega v_{ij}(x)\psi_{ijk,k}(x)\,dx :\ \psi \in \mathcal{C}_0^\infty(\Omega; \mathbb{R}^{m\times p\times3}),\ |\psi(x)| \leq 1 \text{ for all } x \in \Omega\right\} \tag{2.1}$$

and say $v \in BV(\Omega)$ if $\int_\Omega |Dv| < +\infty$. We again assume that $|\psi(x)|$ denotes the square root of the sum of the squares of all the components of $\psi(x)$, which is often called the Frobenius norm. Finally, we define the "planar" variation

$$\int_\Omega |D_P v| = \sup\left\{\sum_{\substack{i=1,\dots,m\\ j=1,\dots,p\\ k=1,2}} \int_\Omega v_{ij}(x)\psi_{ijk,k}(x)\,dx :\ \psi \in \mathcal{C}_0^\infty(\Omega; \mathbb{R}^{m\times p\times2}),\ |\psi(x)| \leq 1 \text{ for all } x \in \Omega\right\}.$$

For a matrix-valued function $v \in L^1(S; \mathbb{R}^{m\times p})$ we similarly define

$$\int_S |Dv| = \sup\left\{\sum_{\substack{i=1,\dots,m\\ j=1,\dots,p\\ k=1,2}} \int_S v_{ij}(x)\psi_{ijk,k}(x)\,dx :\ \psi \in \mathcal{C}_0^\infty(S; \mathbb{R}^{m\times p\times2}),\ |\psi(x)| \leq 1 \text{ for all } x \in S\right\}.$$

We remark that if $v \in BV(\Omega_1)$ is independent of z_3, then, abusing the notation slightly, we have

$$\int_{\Omega_1} |Dv| = \int_{\Omega_1} |D_P v| = \int_S |Dv|.$$

The notation $BV_q(\Omega)$ will denote the space $BV(\Omega) \cap L^q(\Omega)$.

For $A \in \mathbb{R}^{m\times p}$ and $B \in \mathbb{R}^{m\times q}$, we denote by $(A|B) \in \mathbb{R}^{m\times(p+q)}$ the matrix whose first p columns are those of A and whose last q columns are those of B. For $v \in L^1(\Omega_1;\mathbb{R}^{m\times p})$ and $b \in L^1(\Omega_1;\mathbb{R}^m)$, we will use the identity

$$\int_{\Omega_1} |D(v|\sqrt{2}b)| = \int_{\Omega_1} |D(v|b|b)|. \tag{2.2}$$

We will use the following extension of the classical result on the lower semicontinuity of the BV seminorm [18, 20] to functions with fixed trace [10].

Theorem 2.1. *If $w_j, b_j \in BV(\Omega_1)$ for $j \in \mathbb{N}$ and $w, b \in BV(\Omega_1)$ satisfy*

$$\lim_{j\to\infty} \|w_j - w\|_{L^1(\Omega_1)} = 0 \quad \text{and} \quad \lim_{j\to\infty} \|b_j - b\|_{L^1(\Omega_1)} = 0,$$

and $b_j = b_0$ on $\Gamma_1 = \gamma \times (-\frac{1}{2}, \frac{1}{2})$ for fixed $b_0 \in BV(\Omega_1)$, then

$$\int_{\Omega_1} |D_P(w|\sqrt{2}\,b)| + \sqrt{2}\int_{\Gamma_1} |b - b_0| \le \liminf_{j\to\infty} \int_{\Omega_1} |D_P(w_j|\sqrt{2}\,b_j)|.$$

We will also use the following extension of the classical result on the approximation by smooth functions in the BV seminorm [18, 20] to functions with fixed trace [10].

Theorem 2.2. *Let $1 \le q < +\infty$, let $b_0 \in W^{1,q}(S)$ be such that $\nabla b_0 \in BV(S)$, let $b \in BV_q(S)$, and let $w \in BV(S)$. Then there exists a family $\{b_\varepsilon :\ \varepsilon > 0\} \subset W^{1,q}(S)$ with $\nabla b_\varepsilon \in BV(S)$ such that $b_\varepsilon = b_0$ on γ for every $\varepsilon > 0$, and*

$$\lim_{\varepsilon\to 0} \|b_\varepsilon - b\|_{L^q(S)} = 0,$$

$$\lim_{\varepsilon\to 0} \int_S |D(w|\sqrt{2}\,b_\varepsilon)| = \int_S |D(w|\sqrt{2}\,b)| + \sqrt{2}\int_\gamma |b - b_0|.$$

3. The Elastic Energy Density ϕ

We will assume that the energy density $\phi : \mathbb{R}^{3\times3} \times \Omega_1 \to \mathbb{R}$ satisfies the Carathéodory condition [15, 26]

(1) $\phi(F, \hat{z}, z_3)$ is continuous in $(F, z_3) \in \mathbb{R}^{3\times3} \times (-1/2, 1/2)$ for almost every $\hat{z} \in S$,
(2) $\phi(F, \hat{z}, z_3)$ is measurable in $\hat{z} \in S$ for every $(F, z_3) \in \mathbb{R}^{3\times3} \times (-1/2, 1/2)$,

and satisfies the growth condition

$$c_1|F|^p - c_2 \le \phi(F, z) \le c_3(|F|^p + 1) \quad \text{for all } F \in \mathbb{R}^{3\times3} \text{ and } z \in \Omega_1, \tag{3.1}$$

where c_1, c_2, and c_3 are fixed positive constants and $3 < p < +\infty$.

We can obtain this energy density ϕ from a free energy density $\hat{\phi}(F, \theta, c)$, where $\theta(z)$ is a given temperature and $c(z)$ is a given order parameter such as alloy composition, by $\phi(F, z) = \hat{\phi}(F, \theta(z), c(z))$. In what follows, we will usually not denote the explicit dependence of ϕ on z. Notice that ϕ is bounded below and its absolute value satisfies the growth property

$$|\phi(F, z)| \le c_3|F|^p + \max\{c_2, c_3\} \quad \text{for all } F \in \mathbb{R}^{3\times3} \text{ and } z \in \Omega_1.$$

4. The Γ-limit

We now give a definition of Γ-convergence [3, 7, 26] that allows the domain $\tilde{\mathcal{A}}$ of the approximating functionals $\mathcal{F}_h$ to be different than the domain $\mathcal{A}$ of the Γ-limit $\mathcal{F}$.

Definition 4.1. Let $\mathcal{A}$ and $\tilde{\mathcal{A}}$ be spaces such that the convergence of elements of $\tilde{\mathcal{A}}$ to an element of $\mathcal{A}$ is defined. We say that the family of functionals $\{\mathcal{F}_h : \tilde{\mathcal{A}} \to \mathbb{R}\cup\{+\infty\}$ for $h > 0\}$ Γ-converges to $\mathcal{F} : \mathcal{A} \to \mathbb{R}\cup\{+\infty\}$ as $h \to 0$ if the following two conditions are satisfied:

Lower Bound: given any $u \in \mathcal{A}$ and any family $\{u_h \in \tilde{\mathcal{A}} :\ h > 0\}$ such that $u_h \to u$ as $h \to 0$, we have

$$\mathcal{F}(u) \le \liminf_{h\to 0} \mathcal{F}_h(u_h);$$

Upper Bound: given any $u \in \mathcal{A}$, there exists a family $\{u_h \in \tilde{\mathcal{A}} :\ h > 0\}$ such that $u_h \to u$ as $h \to 0$ and

$$\mathcal{F}(u) \geq \limsup_{h\to 0} \mathcal{F}_h(u_h),$$

or equivalently, in view of the lower bound above,

$$\mathcal{F}(u) = \lim_{h\to 0} \mathcal{F}_h(u_h).$$

We note that the first condition above (Lower Bound) guarantees that $\mathcal{F}$ is below the Γ-limit (if it exists), and the second condition (Upper Bound) guarantees that $\mathcal{F}$ is above the Γ-limit (if it exists). If $\mathcal{F}$ satisfies both conditions, then $\mathcal{F}$ is the Γ-limit.

5. Γ-limit of the film model with Dirichlet boundary conditions

In this section, we assume that the film adheres to a rigid material on its lateral surface

$$\Gamma_h = \gamma \times (-h/2,\, h/2),$$

where we assume that $\gamma \neq \emptyset$ is a finite union of connected $\mathcal{C}^{1,1}$ open subsets of ∂S. Let $y_0, b_0 \in W^{1,p}(S;\mathbb{R}^3)$ be such that $\nabla y_0, \nabla b_0 \in BV(S)$ and define the boundary condition

$$\tilde{u}_0(x_1, x_2, x_3) = y_0(x_1, x_2) + b_0(x_1, x_2)x_3 \quad \text{ for } (x_1, x_2, x_3) \in \Omega_h. \tag{5.1}$$

We then define the space $\mathcal{A}_h$ of admissible deformations of the domain Ω_h by

$$\mathcal{A}_h = \left\{\tilde{u} \in W^{1,p}(\Omega_h;\mathbb{R}^3) :\ \nabla\tilde{u} \in BV(\Omega_h),\ \tilde{u} = \tilde{u}_0 \text{ on } \Gamma_h\right\}.$$

We note that due to the growth condition (3.1), we have that

$$\mathcal{A}_h = \left\{\tilde{u} : \Omega_h \to \mathbb{R}^3 :\ \mathcal{E}_h(\tilde{u}) < +\infty,\ \tilde{u} = \tilde{u}_0 \text{ on } \Gamma_h\right\}.$$

Also, since $p > 3$, it follows from the Sobolev embedding theorem [2] that $\mathcal{A}_h \subset \mathcal{C}(\bar{\Omega}_h)$. This ensures that there is no tear in the deformed configurations $\tilde{u}(\Omega_h)$ for $\tilde{u} \in \mathcal{A}_h$.

We are interested in studying the thin film limit of the energies

$$\mathcal{E}_h(\tilde{u}) = \kappa \int_{\Omega_h} |D(\nabla\tilde{u})| + \int_{\Omega_h} \phi(\nabla\tilde{u}(x), x)\, dx, \qquad \tilde{u} \in \mathcal{A}_h, \tag{5.2}$$

where the constant $\kappa > 0$ is a measure of interfacial energy per unit area. We rescale the deformations $\tilde{u} : \Omega_h \to \mathbb{R}^3$ to deformations on a fixed domain of thickness one, $u : \Omega_1 \to \mathbb{R}^3$, via

$$u(z_1, z_2, z_3) = \tilde{u}(z_1, z_2, hz_3) \quad \text{ for } z = (z_1, z_2, z_3) \in \Omega_1, \tag{5.3}$$

and we then study the Γ-convergence as $h \to 0$ of the rescaled energy

$$\mathcal{E}_1^{(h)}(u) = \frac{1}{h}\, \mathcal{E}_h(\tilde{u}) \tag{5.4}$$

for u defined in the space of admissible deformations

$$\mathcal{A}_1 = \left\{u \in W^{1,p}(\Omega_1;\mathbb{R}^3) :\ \nabla u \in BV(\Omega_1),\ u = u_0 \text{ on } \Gamma_1\right\},$$

where u_0 is defined by (5.1) and (5.3) to be

$$u_0(z_1, z_2, z_3) = y_0(z_1, z_2) + b_0(z_1, z_2)hz_3 \quad \text{ for } (z_1, z_2, z_3) \in \Omega_1.$$

We will first define a topology for the convergence of $u_h \in \mathcal{A}_1$ to $(y, b) \in \mathcal{A}_0$ where

$$\mathcal{A}_0 = \{(y, b) \in W^{1,p}(S;\mathbb{R}^3) \times L^p(S;\mathbb{R}^3) :\ \nabla y, b \in BV(S),\ y = y_0 \text{ on } \gamma\}, \tag{5.5}$$

and we will then show that the Γ-limit of $\mathcal{E}_1^{(h)}$ is given by $\mathcal{E}^{(0)}$ where

$$\mathcal{E}^{(0)}(y, b) = \kappa\left[\int_S |D(\nabla y|\sqrt{2}\, b)| + \sqrt{2}\int_\gamma |b - b_0|\right] + \int_S \phi(\nabla y(\hat{z})|b(\hat{z}), \hat{z}, 0)\, d\hat{z}. \tag{5.6}$$

We here use Definition 4.1 with $\mathcal{A} = \mathcal{A}_1$ and $\tilde{\mathcal{A}} = \mathcal{A}_0$. We note that above and in what follows we will often use the notation

$$\int_S \phi(\nabla y|b) = \int_S \phi(\nabla y(\hat{z})|b(\hat{z}), \hat{z}, 0)\, d\hat{z}.$$

In a second approach, we will set $\mathcal{A} = \tilde{\mathcal{A}} = \mathcal{A}_1$ with the topology on the space $\mathcal{A}_1$ given by weak $W^{1,p}$ convergence, and we will prove that a related functional $\mathcal{E}_1^{(0)}$ is the Γ-limit of $\mathcal{E}_1^{(h)}$ as $h \to 0$. The relation between $\mathcal{E}^{(0)}$ and $\mathcal{E}_1^{(0)}$ will become clear.

We now consider the Γ-convergence of $\mathcal{E}_1^{(h)}$ to $\mathcal{E}^{(0)}$ as $h \to 0$. We start by introducing a notion of the convergence of 3-D deformations $\{u_h\} \subset \mathcal{A}_1$ to a 2-D deformation $(y, b) \in \mathcal{A}_0$ as $h \to 0$.

Definition 5.1. We shall say that a family $\{u_h \in \mathcal{A}_1 : \ h > 0\}$ converges to $(y, b) \in \mathcal{A}_0$ if the following conditions are satisfied for $\hat{y}(z_1, z_2, z_3) = y(z_1, z_2)$ and $\hat{b}(z_1, z_2, z_3) = b(z_1, z_2)$:

$$\left.\begin{array}{lllll} u_h \rightharpoonup \hat{y} & \text{in } W^{1,p}(\Omega_1;\mathbb{R}^3) & \text{and} & h^{-1}u_{h,3} \rightharpoonup \hat{b} & \text{in } L^p(\Omega_1;\mathbb{R}^3) \\ u_h \to \hat{y} & \text{in } W^{1,1}(\Omega_1;\mathbb{R}^3) & \text{and} & h^{-1}u_{h,3} \to \hat{b} & \text{in } L^1(\Omega_1;\mathbb{R}^3) \end{array}\right\} \text{ as } h \to 0.$$

We shall use this definition of convergence when proving the Γ-convergence of the functionals $\mathcal{E}_1^{(h)}$ to $\mathcal{E}^{(0)}$ since it allows the compactness property of Lemma 5.1 for sequences of deformations

$$\{u_{h_n} \in \mathcal{A}_1 : n = 1, \dots \text{ and } h_n \to 0 \text{ as } n \to \infty\}$$

with uniformly bounded energy $\mathcal{E}_1^{(h_n)}(u_{h_n}) \le C$ for all $n \ge 1$. This compactness property can then be used with the Γ-convergence of the functionals $\mathcal{E}_1^{(h)}$ to $\mathcal{E}^{(0)}$ to give a proof of the convergence of minimizers of $\mathcal{E}_1^{(h)}$ to minimizers of the Γ-limit $\mathcal{E}^{(0)}$ (see Corollary 5.1 following the proof of Theorem 5.1). We will see from the proof of Theorem 5.1 that $\mathcal{E}^{(0)}$ is also the Γ-limit of $\mathcal{E}_1^{(h)}$ if we use the strong convergence

$$u_h \to \hat{y} \quad \text{in } W^{1,p}(\Omega_1;\mathbb{R}^3) \qquad \text{and} \qquad h^{-1}u_{h,3} \to \hat{b} \quad \text{in } L^p(\Omega_1;\mathbb{R}^3)$$

in Definition 5.1, but we do not have a compactness property for this topology since $BV(\Omega_1)$ is not compactly embedded in $L^p(\Omega_1)$ if $p \ge \frac{3}{2}$ [20].

Lemma 5.1. *Suppose that $\{u_{h_n} \in \mathcal{A}_1 : n = 1, \dots$ and $h_n \to 0$ as $n \to \infty\}$ is a sequence of deformations with uniformly bounded energy $\mathcal{E}_1^{(h_n)}(u_{h_n}) \le C$ for all $n \ge 1$. Then there exists a further subsequence, also denoted by $\{u_{h_n} \in \mathcal{A}_1 : n = 1, \dots\}$, and $(y, b) \in \mathcal{A}_0$ such that $\{u_{h_n} \in \mathcal{A}_1 : n = 1, \dots\}$ converges to $(y, b) \in \mathcal{A}_0$ in the sense of Definition 5.1. We may take a further subsequence such that the convergence is also almost everywhere in Ω_1.*

Proof. We have from the definition of the total variation for matrix valued functions (2.1) that

$$\begin{aligned} &\frac{1}{h_n}\int_{\Omega_{h_n}} |D(\nabla \tilde{u}_{h_n})| \\ &\quad = \sup\Bigg\{ \sum_{\substack{i=1,2,3\\ j,k=1,2}} \int_{\Omega_1} (u_{h_n})_{i,j}\psi_{ijk,k} + \sum_{\substack{i=1,2,3\\ j=1,2}} \int_{\Omega_1} {h_n}^{-1}(u_{h_n})_{i,j}\psi_{ij3,3} + \sum_{\substack{i=1,2,3\\ k=1,2}} \int_{\Omega_1} {h_n}^{-1}(u_{h_n})_{i,3}\psi_{i3k,k} \\ &\qquad\qquad + \sum_{i=1,2,3} \int_{\Omega_1} {h_n}^{-2}(u_{h_n})_{i,3}\psi_{i33,3} :\ \psi \in \mathcal{C}_0^\infty(\Omega_1),\ |\psi(z)| \le 1 \ \text{ for all } z \in \Omega_1 \Bigg\}. \end{aligned} \tag{5.7}$$

Since $\mathcal{E}_1^{(h_n)}(u_{h_n}) \le C$ for all $n \ge 1$, we have by the growth condition (3.1) that

$$\|u_{h_n}\|_{W^{1,p}(\Omega_1;\mathbb{R}^3)} \le C, \qquad \|h_n^{-1}u_{h_n,3}\|_{L^p(\Omega_1;\mathbb{R}^3)} \le C, \tag{5.8}$$

and we have by (5.7) that

$$\begin{aligned}&\int_{\Omega_1} |D(\nabla u_{h_n})| \le C, \qquad \int_{\Omega_1} |D(h_n^{-1} u_{h_n,3})| \le C,\\ &\sup\left\{ \sum_{i=1,2,3} \int_{\Omega_1} h_n^{-2} (u_{h_n})_{i,3} \psi_{i33,3} : \ \psi \in \mathcal{C}_0^\infty(\Omega_1),\ |\psi(z)| \le 1 \ \text{ for all } z \in \Omega_1 \right\} \le C\end{aligned} \tag{5.9}$$

for all $n \ge 1$. It then follows from the compactness of the BV spaces [20] and the trace theorem [2] that there exists $\hat{u} \in W^{1,p}(\Omega_1;\mathbb{R}^3)$ such that $\nabla \hat{u} \in BV(\Omega_1)$ and $\hat{u} = y_0$ on $\gamma \times (-\frac{1}{2}, \frac{1}{2})$ and that there exists $\hat{b} \in BV_p(\Omega_1)$ such that for a further subsequence of $\{u_{h_n}\}$, not relabeled, we have that

$$\left.\begin{array}{lllll} u_{h_n} \rightharpoonup \hat{u} & \text{in } W^{1,p}(\Omega_1;\mathbb{R}^3) & \text{and} & h_n^{-1} u_{h_n,3} \rightharpoonup \hat{b} & \text{in } L^p(\Omega_1;\mathbb{R}^3) \\ u_{h_n} \to \hat{u} & \text{in } W^{1,1}(\Omega_1;\mathbb{R}^3) & \text{and} & h_n^{-1} u_{h_n,3} \to \hat{b} & \text{in } L^1(\Omega_1;\mathbb{R}^3) \end{array}\right\} \text{ as } n \to \infty,$$

and the convergence is also almost everywhere in Ω_1. In addition, from (5.8) and (5.9) it follows that $\hat{u}$ and $\hat{b}$ are independent of z_3, so we can set $y(z_1,z_2) = \hat{u}(z_1,z_2,z_3)$ and $b(z_1,z_2) = \hat{b}(z_1,z_2,z_3)$ to prove the lemma. □

We have the following Γ-convergence theorem:

Theorem 5.1. *The functional $\mathcal{E}^{(0)} : \mathcal{A}_0 \to \mathbb{R}$ is the Γ-limit of the functionals $\mathcal{E}_1^{(h)} : \mathcal{A}_1 \to \mathbb{R}$ with respect to the convergence from Definition 5.1; that is,*

Lower Bound: *given any $(y,b) \in \mathcal{A}_0$ and any family $\{u_h \in \mathcal{A}_1 : \ h > 0\}$ that converges to (y,b), we have*

$$\mathcal{E}^{(0)}(y,b) \le \liminf_{h\to 0} \mathcal{E}_1^{(h)}(u_h);$$

Upper Bound: *given any $(y,b) \in \mathcal{A}_0$, there exists a family $\{u_h \in \mathcal{A}_1 : \ h > 0\}$ that converges to (y,b) such that*

$$\mathcal{E}^{(0)}(y,b) \ge \limsup_{h\to 0} \mathcal{E}_1^{(h)}(u_h),$$

or equivalently, in view of the lower bound above,

$$\mathcal{E}^{(0)}(y,b) = \lim_{h\to 0} \mathcal{E}_1^{(h)}(u_h). \tag{5.10}$$

Proof. Lower Bound. To prove the lower bound, let $(y,b) \in \mathcal{A}_0$ and let $\{u_h \in \mathcal{A}_1 : \ h > 0\}$ converge to (y,b) in the sense of Definition 5.1. Consider a subsequence $\{u_{h_n}\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty} \mathcal{E}_1^{(h_n)}(u_{h_n}) = \liminf_{h\to 0} \mathcal{E}_1^{(h)}(u_h)$$

and such that $\nabla_P u_{h_n} \to \nabla_P \hat{y}$ and $h_n^{-1} u_{h_n,3} \to \hat{b}$ almost everywhere in Ω_1 as $n \to \infty$. It follows from the identity (2.2) and

$$\int_\Omega v_{i,j} \psi_{ij3,3} = \int_\Omega v_{i,3} \psi_{ij3,j} \qquad \text{for all } v \in W^{1,1}(\Omega;\mathbb{R}^3) \text{ and } \psi_{ij3} \in \mathcal{C}_0^\infty(\Omega)$$

that

$$\begin{aligned}
&\int_{\Omega_1} |D_P(\nabla_P u_{h_n}|\sqrt{2}\,h_n^{-1}u_{h_n,3})| \\
&\quad= \int_{\Omega_1} |D_P(\nabla_P u_{h_n}|h_n^{-1}u_{h_n,3}|h_n^{-1}u_{h_n,3})| \\
&\quad= \sup\Bigg\{ \sum_{\substack{i=1,2,3\\ j,k=1,2}} \int_{\Omega_1} (u_{h_n})_{i,j}\psi_{ijk,k} + \sum_{\substack{i=1,2,3\\ j=1,2}} \int_{\Omega_1} {h_n}^{-1}(u_{h_n})_{i,3}\psi_{ij3,j} \\
&\qquad\qquad + \sum_{\substack{i=1,2,3\\ k=1,2}} \int_{\Omega_1} {h_n}^{-1}(u_{h_n})_{i,3}\psi_{i3k,k} :\ \psi \in \mathcal{C}_0^\infty(\Omega_1),\ |\psi(z)| \le 1 \ \text{ for all } z \in \Omega_1 \Bigg\} \\
&\quad\le \sup\Bigg\{ \sum_{\substack{i=1,2,3\\ j,k=1,2}} \int_{\Omega_1} (u_{h_n})_{i,j}\psi_{ijk,k} + \sum_{\substack{i=1,2,3\\ j=1,2}} \int_{\Omega_1} {h_n}^{-1}(u_{h_n})_{i,j}\psi_{ij3,3} + \sum_{\substack{i=1,2,3\\ k=1,2}} \int_{\Omega_1} {h_n}^{-1}(u_{h_n})_{i,3}\psi_{i3k,k} \\
&\qquad\qquad + \sum_{i=1,2,3} \int_{\Omega_1} {h_n}^{-2}(u_{h_n})_{i,3}\psi_{i33,3} :\ \psi \in \mathcal{C}_0^\infty(\Omega_1),\ |\psi(z)| \le 1 \ \text{ for all } z \in \Omega_1 \Bigg\} \\
&\quad= \frac{1}{h_n}\int_{\Omega_{h_n}} |D(\nabla \tilde{u}_{h_n})|.
\end{aligned} \tag{5.11}$$

So, by using Theorem 2.1 on (5.11) and using Fatou's Lemma to control the ϕ term, we obtain that

$$\begin{aligned}
\mathcal{E}^{(0)}(y,b) &= \kappa\left[\int_{\Omega_1}\left|D_P(\nabla_P\hat{y}|\sqrt{2}\,\hat{b})\right| + \sqrt{2}\int_{\Gamma_1}|\hat{b}-b_0|\right] + \int_{\Omega_1}\phi(\nabla_P\hat{y}|\hat{b},\hat{z},0)\,dz \\
&\le \liminf_{n\to\infty}\mathcal{E}_1^{(h_n)}(u_{h_n}) \\
&= \lim_{n\to\infty}\mathcal{E}_1^{(h_n)}(u_{h_n}) \\
&= \liminf_{h\to 0}\mathcal{E}_1^{(h)}(u_h),
\end{aligned}$$

and this establishes the first part of the theorem. We note that above and in what follows we use the convention $z = (\hat{z}, z_3)$ for $\hat{z} \in S$ and $z_3 \in (-1/2, 1/2)$.

Upper Bound. To prove the upper bound, we would like to consider deformations of the form $y(z_1,z_2) + hz_3b(z_1,z_2)$; however, such deformations do not belong to $\mathcal{A}_1$ because b does not belong to $W^{1,p}(S;\mathbb{R}^3)$ and ∇b does not belong to $BV(S)$. We can overcome this problem by using Theorem 2.2: since $b_0 \in W^{1,p}(S;\mathbb{R}^3)$ and $\nabla b_0 \in BV(S)$, there exists a family of functions $b_\varepsilon \in W^{1,p}(S;\mathbb{R}^3)$ with $\nabla b_\varepsilon \in BV(S)$ such that $b_\varepsilon = b_0$ on γ for every $\varepsilon > 0$, $b_\varepsilon \to b$ almost everywhere in S and in $L^p(S)$ as $\varepsilon \to 0$, and

$$\lim_{\varepsilon\to 0}\int_S |D(\nabla y|\sqrt{2}\,b_\varepsilon)| = \int_S |D(\nabla y|\sqrt{2}\,b)| + \sqrt{2}\int_\gamma |b-b_0|. \tag{5.12}$$

We construct the functions

$$u_h^\varepsilon(z_1,z_2,z_3) = y(z_1,z_2) + hz_3b_\varepsilon(z_1,z_2) \in \mathcal{A}_1 \quad \text{for } 0 < h \le 1.$$

Now $\nabla_P u_h^\varepsilon = \nabla_P y + hz_3\nabla_P b_\varepsilon \to \nabla_P y$ in $L^p(\Omega_1)$ and almost everywhere in Ω_1 as $h \to 0$, so we can obtain by using the growth condition (3.1) for ϕ, the Carathéodory property of ϕ given in Section 3, and the dominated convergence theorem that

$$\frac{1}{h}\int_{\Omega_h}\phi(\nabla\tilde{u}_h^\varepsilon(x),x)\,dx = \int_{\Omega_1}\phi(\nabla_P u_h^\varepsilon|h^{-1}u_{h,3}^\varepsilon,\hat{z},hz_3)\,dz = \int_{\Omega_1}\phi(\nabla_P u_h^\varepsilon|b_\varepsilon,\hat{z},hz_3)\,dz \to \int_{\Omega_1}\phi(\nabla_P y|b_\varepsilon,\hat{z},0)\,dz$$

as $h \to 0$. By the same argument,

$$\int_{\Omega_1} \phi(\nabla_P y | b_\varepsilon, \hat{z}, 0)\, dz = \int_S \phi(\nabla y | b_\varepsilon) \to \int_S \phi(\nabla y | b) \quad \text{ as } \varepsilon \to 0,$$

so

$$\lim_{\varepsilon \to 0} \lim_{h \to 0} \frac{1}{h} \int_{\Omega_h} \phi(\nabla \tilde{u}_h^\varepsilon(x), x)\, dx = \int_S \phi(\nabla y | b). \tag{5.13}$$

We now have since b_ε is independent of z_3 that

$$\begin{aligned} \frac{1}{h} \int_{\Omega_h} &|D(\tilde{u}_h^\varepsilon)| \\ &= \sup \Bigg\{ \sum_{\substack{i=1,2,3 \\ j,k=1,2}} \int_{\Omega_1} (u_h^\varepsilon)_{i,j} \psi_{ijk,k} + \sum_{\substack{i=1,2,3 \\ j=1,2}} \int_{\Omega_1} h^{-1} (u_h^\varepsilon)_{i,3} \psi_{ij3,j} + \sum_{\substack{i=1,2,3 \\ k=1,2}} \int_{\Omega_1} h^{-1} (u_h^\varepsilon)_{i,3} \psi_{i3k,k} \\ &\qquad + \sum_{i=1,2,3} \int_{\Omega_1} h^{-2} (u_h^\varepsilon)_{i,3} \psi_{i33,3} : \ \psi \in \mathcal{C}_0^\infty(\Omega),\ |\psi(z)| \le 1 \ \text{ for all } z \in \Omega_1 \Bigg\} \\ &= \sup \Bigg\{ \sum_{\substack{i=1,2,3 \\ j,k=1,2}} \int_{\Omega_1} (u_h^\varepsilon)_{i,j} \psi_{ijk,k} + \sum_{\substack{i=1,2,3 \\ j=1,2}} \int_{\Omega_1} b_\varepsilon \psi_{ij3,j} + \sum_{\substack{i=1,2,3 \\ k=1,2}} \int_{\Omega_1} b_\varepsilon \psi_{i3k,k} : \\ &\qquad \psi \in \mathcal{C}_0^\infty(\Omega_1),\ |\psi(z)| \le 1 \ \text{ for all } z \in \Omega_1 \Bigg\} \\ &= \int_{\Omega_1} \left| D_P(\nabla_P u_h^\varepsilon | \sqrt{2}\, b_\varepsilon) \right|. \end{aligned} \tag{5.14}$$

Since $\nabla b_\varepsilon \in BV(S)$ and y and b_ε are independent of z_3, we have that

$$\lim_{h \to 0} \int_{\Omega_1} \left| D_P(\nabla_P u_h^\varepsilon | \sqrt{2}\, b_\varepsilon) \right| = \lim_{h \to 0} \int_{\Omega_1} |D_P(\nabla_P(y + h z_3 b_\varepsilon) | \sqrt{2}\, b_\varepsilon)| = \int_S |D(\nabla y | \sqrt{2}\, b_\varepsilon)|. \tag{5.15}$$

It then follows from (5.14), (5.15), and (5.12) that

$$\lim_{\varepsilon \to 0} \lim_{h \to 0} \frac{1}{h} \int_{\Omega_h} |D(\tilde{u}_h^\varepsilon)| = \lim_{\varepsilon \to 0} \lim_{h \to 0} \int_{\Omega_1} |D_P(\nabla_P u_h^\varepsilon | \sqrt{2} b_\varepsilon)| = \int_S |D(\nabla y | \sqrt{2}\, b)| + \sqrt{2} \int_\gamma |b - b_0|. \tag{5.16}$$

We can then conclude from (5.13) and (5.16) that

$$\lim_{\varepsilon \to 0} \lim_{h \to 0} \mathcal{E}_1^{(h)}(u_h^\varepsilon) = \mathcal{E}^{(0)}(y, b). \tag{5.17}$$

We note that in view of (5.17), it is clear that for any $\eta > 0$ there exists $\varepsilon > 0$ and $h_0 > 0$ such that

$$|\mathcal{E}_1^{(h)}(u_h^\varepsilon) - \mathcal{E}^{(0)}(y, b)| < \eta \quad \text{ for all } 0 < h \le h_0.$$

□

Corollary 5.1. *For every sequence $\{u_h \in \mathcal{A}_1 : h \to 0\}$ of minimizers of $\mathcal{E}_1^{(h)}$, there exists a subsequence $\{u_{h_n} \in \mathcal{A}_1 : n = 1, \dots$ and $h_n \to 0$ as $n \to \infty\}$ and a minimizer $(y, b) \in \mathcal{A}_0$ of $\mathcal{E}^{(0)}$ such that $\{u_{h_n} \in \mathcal{A}_1 : n = 1, \dots\}$ converges to $(y, b) \in \mathcal{A}_0$ in the sense of Definition 5.1.*

Proof. We first note that $\mathcal{E}_1^{(h)}(u_0)$ is bounded independent of $h > 0$. We can thus prove the existence of minimizers, $u_h \in \mathcal{A}_1$, of the functional $\mathcal{E}_1^{(h)}$ for fixed $h > 0$ by using the bounds (5.8) and (5.9), the compactness and lower-semicontinuity of the BV spaces [20], and Fatou's lemma.

Since $\mathcal{E}_1^{(h)}(u_0)$ is bounded independent of $h > 0$, we have the uniform bound

$$\mathcal{E}_1^{(h)}(u_h) \le \mathcal{E}_1^{(h)}(u_0) \le C.$$

We let $\{u_{h_n} \in \mathcal{A}_1 : n = 1, \dots \text{ and } h_n \to 0 \text{ as } n \to \infty\}$ be a subsequence such that

$$\lim_{n\to\infty} \mathcal{E}_1^{(h_n)}(u_{h_n}) = \liminf_{h\to 0} \mathcal{E}_1^{(h)}(u_h).$$

We can conclude from Lemma 5.1 that there exists a further subsequence (not relabeled), $\{u_{h_n} \in \mathcal{A}_1 : n = 1, \dots \text{ and } h_n \to 0 \text{ as } n \to \infty\}$, and $(y, b) \in \mathcal{A}_0$ such that $\{u_{h_n} \in \mathcal{A}_1 : n = 1, \dots\}$ converges to $(y, b) \in \mathcal{A}_0$ in the sense of Definition 5.1. It follows from the lower bound in Theorem 5.1 that

$$\mathcal{E}^{(0)}(y, b) \le \lim_{n\to\infty} \mathcal{E}_1^{(h_n)}(u_{h_n}) = \liminf_{h\to 0} \mathcal{E}_1^{(h)}(u_h).$$

Since $u_h \in \mathcal{A}_1$ are minimizers of $\mathcal{E}_1^{(h)}$, we can conclude from the upper bound in Theorem 5.1 that $\limsup_{h\to 0} \mathcal{E}_1^{(h)}(u_h) \le \mathcal{E}^{(0)}(y, b)$, so $\lim_{h\to 0} \mathcal{E}_1^{(h)}(u_h)$ exists and

$$\mathcal{E}^{(0)}(y, b) = \lim_{h\to 0} \mathcal{E}_1^{(h)}(u_h).$$

We can now conclude from the upper bound in Theorem 5.1 that $(y, b) \in \mathcal{A}_0$ is a minimizer of $\mathcal{E}^{(0)}$. □

We next address the question of Γ-convergence of $\mathcal{E}_1^{(h)}$ with respect to the weak convergence in $\mathcal{A}_1$. We start by considering the problem of minimizing $\mathcal{E}^{(0)}(y, b)$ with respect to b.

Lemma 5.2. *Let $y \in W^{1,p}(S; \mathbb{R}^3)$ be such that $\nabla y \in BV(S)$ and $y = y_0$ on γ. Then*

$$\inf_{b \in BV_p(S;\mathbb{R}^3)} \mathcal{E}^{(0)}(y, b) = \inf_{\substack{b \in BV_p(S;\mathbb{R}^3) \\ b = b_0 \text{ on } \gamma}} \mathcal{E}^{(0)}(y, b). \tag{5.18}$$

Proof. It is clear that the left-hand side is less than or equal to the right-hand side since the infimum is taken over a larger space.

To show the opposite inequality, it is enough to show that for any $b \in BV_p(S; \mathbb{R}^3)$ the energy $\mathcal{E}^{(0)}(y, b)$ can be arbitrarily closely approximated by energies $\mathcal{E}^{(0)}(y, b)$ with $b \in BV_p(S; \mathbb{R}^3)$ such that $b = b_0$ on γ. However, this follows by applying Theorem 2.2 with $q = p$ and showing that the elastic energy term $\int_S \phi(\nabla y|b_\varepsilon)$ again converges to $\int_S \phi(\nabla y|b)$ as in the proof of the second part of Theorem 5.1. □

We next have that the infimum on the left-hand side of (5.18) in Lemma 5.2 is attained for any y.

Lemma 5.3. *Let $y \in W^{1,p}(S; \mathbb{R}^3)$ be such that $\nabla y \in BV(S)$ and $y = y_0$ on γ. Then there exists a function $\tilde{b} \in BV_p(S; \mathbb{R}^3)$ such that*

$$\mathcal{E}^{(0)}(y, \tilde{b}) = \inf_{b \in BV_p(S;\mathbb{R}^3)} \mathcal{E}^{(0)}(y, b).$$

Proof. Since $\mathcal{E}^{(0)}$ is bounded below, we can consider a minimizing sequence $\{b_j\}_{j=1}^\infty \subset BV_p(S; \mathbb{R}^3)$; in view of Lemma 5.2, we can also assume that $b_j = b_0$ on γ for all $j \in \mathbb{N}$. Since the variations of the b_j and their L^p-norms (and thus also the L^1-norms) lie in a compact subset of $\mathbb{R}$, we can use the compactness of $BV(S; \mathbb{R}^3)$ [20] and retrieve a subsequence, not relabeled, which converges to a function $\tilde{b} \in BV_p(S; \mathbb{R}^3)$ strongly in $L^1(S; \mathbb{R}^3)$, weakly in $L^p(S; \mathbb{R}^3)$, and almost everywhere in S. In addition, by applying Theorem 2.1, we have

$$\int_S |D(\nabla y|\sqrt{2}\,\tilde{b})| + \sqrt{2} \int_\gamma |\tilde{b} - b_0| \le \liminf_{j\to\infty} \int_S |D(\nabla y|\sqrt{2}\,b_j)|.$$

Similarly, applying Fatou's Lemma to $\phi(\nabla y|b_j)$ gives

$$\int_S \phi(\nabla y|\tilde{b}) \le \liminf_{j\to\infty} \int_S \phi(\nabla y|b_j),$$

and therefore

$$\begin{aligned} \mathcal{E}^{(0)}(y, \tilde{b}) &\le \liminf_{j\to\infty} \mathcal{E}^{(0)}(y, b_j) \\ &= \inf_{b \in BV_p(S;\mathbb{R}^3)} \mathcal{E}^{(0)}(y, b). \end{aligned}$$

□

We are now in the position to find the Γ-limit of the functionals $\mathcal{E}_1^{(h)} = \frac{1}{h}\,\mathcal{E}_h(\tilde{u})$. Given a continuous $u \in \mathcal{A}_1$, we can define the deformation of the midplane

$$u_M(z_1, z_2) = u(z_1, z_2, 0)$$

and a functional

$$\mathcal{E}_1^{(0)}(u) = \begin{cases} \min\limits_{b \in BV_p(S;\mathbb{R}^3)} \mathcal{E}^{(0)}(u_M, b) & \text{if } u_{,3} = 0 \text{ a.e. in } \Omega_1, \\ +\infty & \text{otherwise.} \end{cases}$$

In what follows, C will denote a generic positive constant independent of h, which can change from line to line.

Theorem 5.2. *The functional $\mathcal{E}_1^{(0)} : \mathcal{A}_1 \to \mathbb{R} \cup \{+\infty\}$ is the Γ-limit of the functionals $\mathcal{E}_1^{(h)} : \mathcal{A}_1 \to \mathbb{R}$ as $h \to 0$ with respect to the weak $W^{1,p}(\Omega_1;\mathbb{R}^3)$ convergence in $\mathcal{A}_1$; that is,*

Lower Bound: *given any $u \in \mathcal{A}_1$ and any family $\{u_h \in \mathcal{A}_1 :\ h > 0\}$ such that $u_h \rightharpoonup u$ in $W^{1,p}(\Omega_1;\mathbb{R}^3)$ as $h \to 0$, we have*

$$\mathcal{E}_1^{(0)}(u) \le \liminf_{h\to 0} \mathcal{E}_1^{(h)}(u_h);$$

Upper Bound: *given any $u \in \mathcal{A}_1$, there exists a family $\{u_h \in \mathcal{A}_1 :\ h > 0\}$ such that $u_h \rightharpoonup u$ in $W^{1,p}(\Omega_1;\mathbb{R}^3)$ as $h \to 0$ and*

$$\mathcal{E}_1^{(0)}(u) \ge \limsup_{h\to 0} \mathcal{E}_1^{(h)}(u_h),$$

or equivalently, in view of the lower bound above,

$$\mathcal{E}_1^{(0)}(u) = \lim_{h\to 0} \mathcal{E}_1^{(h)}(u_h).$$

Proof. Lower Bound. Let $u \in \mathcal{A}_1$ and let $\{u_h :\ h > 0\} \subset \mathcal{A}_1$ be such that $u_h \rightharpoonup u$ in $W^{1,p}(\Omega_1)$ as $h \to 0$. If $\liminf_{h\to 0} \mathcal{E}_1^{(h)}(u_h) = +\infty$, then

$$\mathcal{E}_1^{(0)}(u) \le \liminf_{h\to 0} \mathcal{E}_1^{(h)}(u_h)$$

is trivially satisfied.

On the other hand, if $\liminf_{h\to 0} \mathcal{E}_1^{(h)}(u_h) < +\infty$, then we can first consider a subsequence $\{u_{h_n}\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \mathcal{E}_1^{(h_n)}(u_{h_n}) = \liminf_{h\to 0} \mathcal{E}_1^{(h)}(u_h).$$

Since then $\mathcal{E}_1^{(h_n)}(u_{h_n}) \le C$ for all $n \ge 1$, we have by Lemma 5.1 that there exists $b \in BV_p(\Omega_1;\mathbb{R}^3)$ such that for a further subsequence of $\{u_{h_n}\}$, not relabeled, we have that

$$\left.\begin{array}{llll} u_{h_n} \rightharpoonup u & \text{in } W^{1,p}(\Omega_1;\mathbb{R}^3) & \text{and} \quad h_n^{-1} u_{h_n,3} \rightharpoonup b & \text{in } L^p(\Omega_1;\mathbb{R}^3) \\ u_{h_n} \to u & \text{in } W^{1,1}(\Omega_1;\mathbb{R}^3) & \text{and} \quad h_n^{-1} u_{h_n,3} \to b & \text{in } L^1(\Omega_1;\mathbb{R}^3) \end{array}\right\} \text{ as } n \to \infty,$$

and the convergence is also almost everywhere in Ω_1. It also follows from Lemma 5.1 that u and b are independent of z_3. Therefore, by the definition of $\mathcal{E}_1^{(0)}(u)$, we have for $u_M(z_1, z_2) = u(z_1, z_2, 0)$ and $b_M(z_1, z_2) = b(z_1, z_2, 0)$ that

$$\mathcal{E}_1^{(0)}(u) \le \mathcal{E}^{(0)}(u_M, b_M). \tag{5.19}$$

Using (5.11), Theorem 2.1, and Fatou's Lemma to control the ϕ term, we have that

$$\begin{aligned} \mathcal{E}^{(0)}(u_M, b_M) &= \kappa \left[\int_{\Omega_1} |D_P(\nabla_P u|\sqrt{2}\, b)| + \sqrt{2} \int_{\Gamma_1} |b - b_0| \right] + \int_{\Omega_1} \phi(\nabla_P u|b, \hat{z}, 0)\, d\hat{z} \\ &\le \liminf_{n\to\infty} \mathcal{E}_1^{(h_n)}(u_{h_n}) \\ &= \lim_{n\to\infty} \mathcal{E}_1^{(h_n)}(u_{h_n}) \\ &= \liminf_{h\to 0} \mathcal{E}_1^{(h)}(u_h). \end{aligned}$$

Combining the above result with (5.19) completes the first part of the proof.

Upper Bound. If $u \in \mathcal{A}_1$ is not independent of z_3, then $\mathcal{E}_1^{(0)}(u) = +\infty$ and

$$\mathcal{E}_1^{(0)}(u) \geq \limsup_{h \to 0} \mathcal{E}_1^{(h)}(u_h)$$

holds for any family $\{u_h \in \mathcal{A}_1 : \ h > 0\}$ such that $u_h \rightharpoonup u$ in $W^{1,p}(\Omega_1)$ as $h \to 0$.

On the other hand, if $u_{,3} = 0$ a.e. in Ω_1, then by Lemma 5.3 there exists $\tilde{b} \in BV_p(S;\mathbb{R}^3)$ such that $\mathcal{E}_1^{(0)}(u) = \mathcal{E}^{(0)}(u_M, \tilde{b})$. Using the upper bound of Theorem 5.1, there exists a family $\{u_h \in \mathcal{A}_1 : \ h > 0\}$ such that

$$u_h \rightharpoonup u \quad \text{ in } W^{1,p}(\Omega_1) \text{ as } h \to 0$$

and

$$\begin{aligned}\lim_{h \to 0} \mathcal{E}_1^{(h)}(u_h) &= \mathcal{E}^{(0)}(u_M, \tilde{b}) \\ &= \mathcal{E}_1^{(0)}(u).\end{aligned}$$

□

We note that the family $\{u_h \in \mathcal{A}_1 : \ h > 0\}$ constructed for the upper bound in Theorem 5.2 actually converges strongly, that is, $u_h \to u$ in $W^{1,p}(\Omega_1)$ as $h \to 0$. We thus have that the functional $\mathcal{E}_1^{(0)} : \mathcal{A}_1 \to \mathbb{R} \cup \{+\infty\}$ is also the Γ-limit of the functionals $\mathcal{E}_1^{(h)} : \mathcal{A}_1 \to \mathbb{R}$ as $h \to 0$ with respect to the strong $W^{1,p}(\Omega_1;\mathbb{R}^3)$ convergence in $\mathcal{A}_1$.

We can obtain the following result on the convergence of minimizers of $\mathcal{E}_1^{(h)}$ to minimizers of $\mathcal{E}_1^{(0)}$ by an argument analogous to that of Corollary 5.1.

Corollary 5.2. *For every sequence $\{u_h \in \mathcal{A}_1 : h \to 0\}$ of minimizers of $\mathcal{E}_1^{(h)}$, there exists a subsequence $\{u_{h_n} \in \mathcal{A}_1 : n = 1, \ldots$ and $h_n \to 0$ as $n \to \infty\}$ and a minimizer $u \in \mathcal{A}_1$ of $\mathcal{E}_1^{(0)}$ such that $\{u_{h_n} \in \mathcal{A}_1 : n = 1, \ldots\}$ converges to $u \in \mathcal{A}_1$ with respect to weak $W^{1,p}(\Omega_1;\mathbb{R}^3)$ convergence in $\mathcal{A}_1$.*

6. Γ-limit of the Dead-Loaded Film Model

We now assume that the film is subject to a dead load Tn on its boundary, $\partial\Omega_h$, with unit exterior normal n where $T \in \mathbb{R}^{3\times 3}$ is independent of $x \in \Omega_h$. In this case, the energy of the three-dimensional thin film is given by

$$\begin{aligned}\hat{\mathcal{E}}_h(\tilde{u}) &= \kappa \int_{\Omega_h} |D(\nabla\tilde{u})| + \int_{\Omega_h} \phi(\nabla\tilde{u}(x), x)\, dx - \int_{\partial\Omega_h} (Tn)\cdot\tilde{u} \\ &= \kappa \int_{\Omega_h} |D(\nabla\tilde{u})| + \int_{\Omega_h} \phi(\nabla\tilde{u}(x), x)\, dx - \int_{\Omega_h} T\cdot\nabla\tilde{u}.\end{aligned}$$

If the elastic energy density ϕ satisfies the growth condition (3.1), then we can define

$$\hat{\phi}(F, x) = \phi(F, x) - T \cdot F$$

and $\hat{\phi}$ still satisfies (3.1) for some different positive constants, which we still denote by c_1, c_2, and c_3.

In this case, we define the space $\hat{\mathcal{A}}_h$ of admissible deformations of the domain Ω_h by

$$\hat{\mathcal{A}}_h = \left\{\tilde{u} \in W^{1,p}(\Omega_h;\mathbb{R}^3) : \ \nabla\tilde{u} \in BV(\Omega_h),\ \int_{\Omega_h} \tilde{u} = 0\right\}.$$

The energies of the deformations $\tilde{u} \in \hat{\mathcal{A}}_h$ of films are again given by (5.2) with $\phi(F)$ replaced by $\phi(F) - T\cdot F$. As before, due to the growth condition (3.1), we have

$$\hat{\mathcal{A}}_h = \left\{\tilde{u} : \Omega_h \to \mathbb{R}^3 : \ \hat{\mathcal{E}}_h(\tilde{u}) < +\infty,\ \int_{\Omega_h} \tilde{u} = 0\right\} \subset \mathcal{C}(\hat{\Omega}_h).$$

The proof of the convergence to a Γ-limit for the problem of the dead-loaded film is similar to the proof for the film constrained on part of the boundary. We start with the rescaled energy $\hat{\mathcal{E}}_1^{(h)} : \hat{\mathcal{A}}_1 \to \mathbb{R}$ (defined by (5.2)–(5.4)), where the space of admissible deformations, $\hat{\mathcal{A}}_1$, is defined by

$$\hat{\mathcal{A}}_1 = \left\{ u \in W^{1,p}(\Omega_1;\mathbb{R}^3) : \ \nabla u \in BV(\Omega_1), \int_{\Omega_1} u = 0 \right\}.$$

We then show that the Γ-limit of $\hat{\mathcal{E}}_1^{(h)} : \hat{\mathcal{A}}_1 \to \mathbb{R}$ is given by

$$\hat{\mathcal{E}}^{(0)}(y,b) = \kappa \int_S |D(\nabla y|\sqrt{2}\, b)| + \int_S \phi(\nabla y(\hat{z})|b(\hat{z}),\hat{z},0)\, d\hat{z} - \int_S T\cdot(\nabla y|b) \quad \text{for } (y,b) \in \hat{\mathcal{A}}_0,$$

where the space of admissible deformations is given by

$$\hat{\mathcal{A}}_0 = \{(y,b) \in W^{1,p}(S;\mathbb{R}^3) \times L^p(S;\mathbb{R}^3) : \ \nabla y, b \in BV(S), \ \int_S y = 0\}.$$

The proof of the following compactness result for sequences $\{u_{h_n} \in \hat{\mathcal{A}}_1 : n = 1,\dots \text{ and } h_n \to 0 \text{ as } n \to \infty\}$ is analogous to that of Lemma 5.1.

Lemma 6.1. *Suppose that* $\{u_{h_n} \in \hat{\mathcal{A}}_1 : n = 1,\dots$ *and* $h_n \to 0$ *as* $n \to \infty\}$ *is a sequence of deformations with uniformly bounded energy* $\hat{\mathcal{E}}_1^{(h_n)}(u_{h_n}) \le C$ *for all* $n \ge 1$*. Then there exists a further subsequence, also denoted by* $\{u_{h_n} \in \hat{\mathcal{A}}_1 : n = 1,\dots\}$*, and* $(y,b) \in \hat{\mathcal{A}}_0$ *such that* $\{u_{h_n} \in \hat{\mathcal{A}}_1 : n = 1,\dots\}$ *converges to* $(y,b) \in \hat{\mathcal{A}}_0$ *in the sense of Definition 5.1 (with the spaces* $\mathcal{A}_1$ *and* $\mathcal{A}_0$ *replaced by* $\hat{\mathcal{A}}_1$ *and* $\hat{\mathcal{A}}_0$*, respectively). We may take a further subsequence such that the convergence is also almost everywhere in* Ω_1*.*

We have the following Γ-convergence theorem.

Theorem 6.1. *The functional* $\hat{\mathcal{E}}^{(0)} : \hat{\mathcal{A}}_0 \to \mathbb{R}$ *is the* Γ*-limit of the functionals* $\hat{\mathcal{E}}_1^{(h)} : \hat{\mathcal{A}}_1 \to \mathbb{R}$ *with respect to the convergence from Definition 5.1.*

Proof. Lower Bound. Let $(y,b) \in \hat{\mathcal{A}}_0$ and let $\{u_h \in \hat{\mathcal{A}}_1 : \ h > 0\}$ converge to (y,b) in the sense of Definition 5.1. Consider a subsequence $\{u_{h_n}\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \hat{\mathcal{E}}_1^{(h_n)}(u_{h_n}) = \liminf_{h\to 0} \hat{\mathcal{E}}_1^{(h)}(u_h)$$

and such that $\nabla_P u_{h_n} \to \nabla_P \hat{y}$ and $h_n^{-1} u_{h_n,3} \to \hat{b}$ almost everywhere in Ω_1 as $n \to \infty$. Using (5.11), the lower semicontinuity of the total variation, and Fatou's Lemma to control the ϕ term, we have that

$$\begin{aligned}
\hat{\mathcal{E}}^{(0)}(y,b) &= \kappa \int_{\Omega_1} |D_P(\nabla_P \hat{y}|\sqrt{2}\,\hat{b})| + \int_{\Omega_1} \phi(\nabla_P \hat{y}|\hat{b},\hat{z},0)\, dz - \int_S T\cdot(\nabla_P\hat{y}|\hat{b}) \\
&\le \liminf_{n\to\infty} \hat{\mathcal{E}}_1^{(h_n)}(u_{h_n}) \\
&= \lim_{n\to\infty} \hat{\mathcal{E}}_1^{(h_n)}(u_{h_n}) \\
&= \liminf_{h\to 0} \hat{\mathcal{E}}_1^{(h)}(u_h),
\end{aligned}$$

which establishes the first part of the theorem.

Upper Bound. To prove the upper bound, one should again consider deformations of the form $y(z_1,z_2) + hz_3 b(z_1,z_2)$; as before, such deformations do not belong to $\hat{\mathcal{A}}_1$, because b does not belong to $W^{1,p}(S;\mathbb{R}^3)$. However, we can find a family of functions $b_\varepsilon \in \mathcal{C}^\infty(\hat{S};\mathbb{R}^3) \subset W^{1,p}(S;\mathbb{R}^3)$ such that $b_\varepsilon \to b$ almost everywhere in S and in $L^p(S)$ as $\varepsilon \to 0$ [14], and

$$\lim_{\varepsilon\to 0} \int_S |D(\nabla y|\sqrt{2}\, b_\varepsilon)| = \int_S |D(\nabla y|\sqrt{2}\, b)|. \tag{6.1}$$

Consider now the functions

$$w_h^\varepsilon(z_1,z_2,z_3) = y(z_1,z_2) + hz_3 b_\varepsilon(z_1,z_2) hz_3 \in \mathcal{A}_1 \quad \text{for } 0 < h \le 1,$$

and their mean-zero translations

$$u_h^\varepsilon = w_h^\varepsilon - \frac{1}{|\Omega_1|}\int_{\Omega_1} w_h^\varepsilon \, dz \in \hat{\mathcal{A}}_1.$$

We can now apply the same argument as in the proof of the upper bound in Theorem 5.1 to conclude that

$$\lim_{\varepsilon\to 0}\lim_{h\to 0} \hat{\mathcal{E}}_1^{(h)}(u_h^\varepsilon) = \hat{\mathcal{E}}^{(0)}(y,b),$$

from which it is clear that for any $\eta > 0$ there exists $\varepsilon > 0$ and $h_0 > 0$ such that

$$|\hat{\mathcal{E}}_1^{(h)}(u_h^\varepsilon) - \hat{\mathcal{E}}^{(0)}(y,b)| < \eta \quad \text{for all } 0 < h \le h_0.$$

□

We can obtain a result on the convergence of minimizers of $\mathcal{E}_1^{(h)}(u)$ to minimizers of $\mathcal{E}^{(0)}(y,b)$ by an argument analogous to that of Corollary 5.2.

The Γ-limit of $\hat{\mathcal{E}}_1^{(h)} : \hat{\mathcal{A}}_1 \to \mathbb{R}$ can again be obtained by minimizing out b in the energy $\hat{\mathcal{E}}^{(0)}(y,b)$. The existence of a minimizing $\tilde{b}$ can be shown by using the direct method of the calculus of variations as in Lemma 5.3.

Lemma 6.2. *Let $y \in W^{1,p}(S;\mathbb{R}^3)$ be such that $\nabla y \in BV(S)$ and $\int_S y = 0$. Then there exists a function $\tilde{b} \in BV_p(S;\mathbb{R}^3)$ such that*

$$\hat{\mathcal{E}}^{(0)}(y,\tilde{b}) = \inf_{b\in BV_p(S;\mathbb{R}^3)} \hat{\mathcal{E}}^{(0)}(y,b).$$

Proof. Since $\hat{\mathcal{E}}^{(0)}$ is bounded below, we can consider a minimizing sequence $\{b_j\}_{j=1}^\infty \subset BV_p(S;\mathbb{R}^3)$. Since the variations of the b_j and their L^p-norms (and thus also the L^1-norms) lie in a compact subset of $\mathbb{R}$, we can use the compactness of $BV(S;\mathbb{R}^3)$ and retrieve a subsequence, not relabeled, which converges to a function $\tilde{b} \in BV_p(S;\mathbb{R}^3)$ strongly in $L^1(S;\mathbb{R}^3)$, weakly in $L^p(S;\mathbb{R}^3)$, and almost everywhere in S. From the lower semicontinuity of the total variation, we have

$$\int_S |D(\nabla y|\sqrt{2}\,\tilde{b}| \le \liminf_{j\to\infty}\int_S |D(\nabla y|\sqrt{2}\,b_j|.$$

Similarly, applying Fatou's Lemma to $\phi(\nabla y|b_j)$ gives

$$\int_S \left[\phi(\nabla y|\tilde{b}) - T\cdot(\nabla y|\tilde{b})\right] \le \liminf_{j\to\infty}\int_S \left[\phi(\nabla y|b_j) - T\cdot(\nabla y|b_j)\right],$$

and therefore

$$\begin{aligned}\hat{\mathcal{E}}^{(0)}(y,\tilde{b}) &\le \liminf_{j\to\infty}\hat{\mathcal{E}}^{(0)}(y,b_j)\\ &= \inf_{b\in BV_p(S;\mathbb{R}^3)} \hat{\mathcal{E}}^{(0)}(y,\tilde{b}).\end{aligned}$$

□

Next, we define a functional

$$\hat{\mathcal{E}}_1^{(0)}(u) = \begin{cases} \min\limits_{b\in BV_p(S;\mathbb{R}^3)} \hat{\mathcal{E}}^{(0)}(u_M,b) & \text{if } u_{,3} = 0 \text{ a.e. in } \Omega_1,\\ +\infty & \text{otherwise.}\end{cases}$$

Theorem 6.2. *The functional $\hat{\mathcal{E}}_1^{(0)} : \hat{\mathcal{A}}_1 \to \mathbb{R}\cup\{+\infty\}$ is the Γ-limit of the functionals $\hat{\mathcal{E}}_1^{(h)} : \hat{\mathcal{A}}_1 \to \mathbb{R}$ as $h \to 0$ with respect to the weak $W^{1,p}(\Omega_1;\mathbb{R}^3)$ convergence in $\hat{\mathcal{A}}_1$.*

Proof. Lower Bound. The proof is similar to the proof of Theorem 5.2. Let $u \in \hat{\mathcal{A}}_1$ and let $\{u_h : h > 0\} \subset \hat{\mathcal{A}}_1$ be such that $u_h \rightharpoonup u$ in $W^{1,p}(\Omega_1)$ as $h \to 0$. If $\liminf_{h\to 0}\hat{\mathcal{E}}_1^{(h)}(u_h) = +\infty$, then

$$\hat{\mathcal{E}}_1^{(0)}(u) \le \liminf_{h\to 0}\hat{\mathcal{E}}_1^{(h)}(u_h)$$

is trivially satisfied.

On the other hand, if $\liminf_{h\to 0} \hat{\mathcal{E}}_1^{(h)}(u_h) < +\infty$, then we can first consider a subsequence $\{u_{h_n}\}_{n=1}^{\infty} \subset \hat{\mathcal{A}}_1$ such that

$$\lim_{n\to\infty} \hat{\mathcal{E}}_1^{(h_n)}(u_{h_n}) = \liminf_{h\to 0} \hat{\mathcal{E}}_1^{(h)}(u_h).$$

Since then $\hat{\mathcal{E}}_1^{(h_n)}(u_{h_n}) \le C$ for all $n \ge 1$, we have by Lemma 6.1 that there exists $b \in BV_p(\Omega_1;\mathbb{R}^3)$ such that for a further subsequence of $\{u_{h_n}\}$, not relabeled, we have that

$$\left.\begin{array}{llll} u_{h_n} \rightharpoonup u & \text{in } W^{1,p}(\Omega_1;\mathbb{R}^3) & \text{and} \quad h_n^{-1}u_{h_n,3} \rightharpoonup b & \text{in } L^p(\Omega_1;\mathbb{R}^3) \\ u_{h_n} \to u & \text{in } W^{1,1}(\Omega_1;\mathbb{R}^3) & \text{and} \quad h_n^{-1}u_{h_n,3} \to b & \text{in } L^1(\Omega_1;\mathbb{R}^3) \end{array}\right\} \text{ as } n\to\infty,$$

and the convergence is also almost everywhere in Ω_1. It also follows from Lemma 6.1 that u and b are independent of z_3. Therefore, we have for $u_M(z_1,z_2) = u(z_1,z_2,0)$ and $b_M(z_1,z_2) = b(z_1,z_2,0)$ that

$$\hat{\mathcal{E}}_1^{(0)}(u) \le \hat{\mathcal{E}}^{(0)}(u_M, b_M). \tag{6.2}$$

Using (5.11), the lower semicontinuity of the total variation, and Fatou's Lemma to control the ϕ term, we have that

$$\begin{aligned} \hat{\mathcal{E}}^{(0)}(u_M,b_M) &= \kappa \int_{\Omega_1} |D_P(\nabla_P u|\sqrt{2}\, b)| + \int_{\Omega_1} \phi(\nabla_P u|b,\hat{z},0)\,dz - \int_{\Omega_1} T\cdot(\nabla_P u|b) \\ &\le \liminf_{n\to\infty} \hat{\mathcal{E}}_1^{(h_n)}(u_{h_n}) \\ &= \lim_{n\to\infty} \hat{\mathcal{E}}_1^{(h_n)}(u_{h_n}) \\ &= \liminf_{h\to 0} \hat{\mathcal{E}}_1^{(h)}(u_h). \end{aligned}$$

Combining the above result with (6.2) completes the first part of the proof.

Upper Bound. If $u \in \hat{\mathcal{A}}_1$ is not independent of z_3, then $\hat{\mathcal{E}}_1^{(0)}(u) = +\infty$; taking $u_h = u$ for all $h > 0$ produces a family in $\hat{\mathcal{A}}_1$ such that $\lim_{h\to 0} \hat{\mathcal{E}}_1^{(h)}(u_h) = \hat{\mathcal{E}}_1^{(0)}(u) = +\infty$.

On the other hand, if $u_{,3} = 0$ a.e. in Ω_1, then by Lemma 6.2 there exists $\tilde{b} \in BV_p(S;\mathbb{R}^3)$ such that $\hat{\mathcal{E}}_1^{(0)}(u) = \hat{\mathcal{E}}^{(0)}(u_M,\tilde{b})$. Using the upper bound of Theorem 6.1, there exists a family $\{u_h \in \hat{\mathcal{A}}_1 : \ h > 0\}$ such that

$$u_h \rightharpoonup u \quad \text{ in } W^{1,p}(\Omega_1) \text{ as } h \to 0$$

and

$$\begin{aligned} \lim_{h\to 0} \hat{\mathcal{E}}_1^{(h)}(u_h) &= \hat{\mathcal{E}}^{(0)}(u_M,\tilde{b}) \\ &= \hat{\mathcal{E}}_1^{(0)}(u). \end{aligned}$$

□

We note that we can obtain a result on the convergence of minimizers of $\mathcal{E}_1^{(h)}$ to minimizers of $\mathcal{E}_1^{(0)}$ by an argument analogous to that of Corollary 5.2.

References

[1] E. Acerbi, G. Buttazzo, and D. Percivale. A variational definition for the strain energy of an elastic string. *Journal of Elasticity*, 25:137–148, 1991.

[2] Robert Adams. *Sobolev Spaces*. Academic Press, New York, 1975.

[3] G. Anzelloti, S. Baldo, and D. Percivale. Dimension reduction in variation problems, asymptotic development in Γ-convergence and thin structures in elasticity. *Asympt. Anal.*, 9:61–100, 1994.

[4] John M. Ball and Richard D. James. Fine phase mixtures as minimizers of energy. *Arch. Rational Mech. Anal.*, 100(1):13–52, 1987.

[5] Patricia Bauman and Daniel Phillips. A nonconvex variational problem related to change of phase. *Appl. Math. Optim.*, 21(2):113–138, 1990.

[6] Kaushik Bhattacharya and Richard D. James. A theory of thin films of martensitic materials with applications to microactuators. *J. Mech. Phys. Solids*, 47(3):531–576, 1999.

[7] Andrea Braides. *Γ-convergence for Beginners*. Oxford University Press, Oxford, 2002.

[8] Pavel Bělík, Tim Brule, and Mitchell Luskin. On the numerical modeling of deformations of pressurized martensitic thin films. *Mathematical Modelling and Numerical Analysis*, 35:525–548, 2001.
[9] Pavel Bělík and Mitchell Luskin. A computational model for the indentation and phase transformation of a martensitic thin film. *J. Mech. Phys. Solids*, 50:1789–1815, 2002.
[10] Pavel Bělík and Mitchell Luskin. A total-variation surface energy model for thin films of martensitic crystals. *Interfaces Free Bound.*, 4(1):71–88, 2002.
[11] Pavel Bělík and Mitchell Luskin. A computational model for martensitic thin films with compositional fluctuation. *Mathematical Models & Methods in Applied Sciences*, 14:1585–1598, 2004.
[12] Pavel Bělík and Mitchell Luskin. Computational modeling of softening in a structural phase transformation. *Multiscale Modeling & Simulation*, 3:764–781, 2004.
[13] Pavel Bělík and Mitchell Luskin. A finite element model for martensitic thin films. Manuscript, 2004.
[14] E. Casas, K. Kunisch, and C. Pola. Regularization by functions of bounded variation and applications to image enhancement. *Appl. Math. Optim.*, 40(2):229–257, 1999.
[15] Bernard Dacorogna. *Direct methods in the calculus of variations*. Springer-Verlag, Berlin, 1989.
[16] J.W. Dong, J.Q. Xie, J. Lu, C. Adelmann, C.J. Palmstrøm, J. Cui, Q. Pan, T.W. Shield, R.D. James, and S. McKernan. Shape memory and ferromagnetic shape memory effects in single-crystal Ni_2MnGa thin films. *J. Appl. Phys.*, 2004.
[17] H. Le Dret and A. Raoult. The nonlinear membrane model as variational limit of nonlinear three-dimensional elasticity. *J. Math. Pures Appl.*, 73:549–578, 1995.
[18] Lawrence C. Evans and Ronald F. Gariepy. *Measure Theory and Fine Properties of Functions*. CRC Press, Boca Raton, FL, 1992.
[19] I. Fonseca and G. Francfort. 3D-2D asymptotic analysis of an optimal design problem for thin films. *J. Reine Angew. Math*, 505:173–202, 1998.
[20] Enrico Giusti. *Minimal Surfaces and Functions of Bounded Variation*. Birkhäuser Verlag, Basel-Boston, Mass., 1984.
[21] Sivan Kartha, James A. Krumhansl, James P. Sethna, and Lisa K. Wickham. Disorder-driven pretransitional tweed in martensitic transformations. *Phys. Rev. B*, 52:803–822, 1995.
[22] P. Krulevitch, A. P. Lee, P. B. Ramsey, J. C. Trevino, J. Hamilton, and M. A. Northrup. Thin film shape memory microactuators. *J. MEMS*, 5:270–282, 1996.
[23] Martin Kružík. Numerical approach to double well problems. *SIAM J. Numer. Anal.*, 35(5):1833–1849, 1998.
[24] Bo Li. Finite element analysis of a class of stress-free martensitic microstructures. *Math. Comp.*, 72(244):1675–1688 (electronic), 2003.
[25] Mitchell Luskin. On the computation of crystalline microstructure. *Acta Numer.*, 5:191–257, 1996.
[26] Gianni Dal Maso. *An introduction to Γ-convergence*. Birkhäuser, 1993.
[27] Luciano Modica and Stefano Mortola. Il limite nella Γ-convergenza di una famiglia di funzionali ellittici. *Boll. Un. Mat. Ital. A (5)*, 14(3):526–529, 1977.
[28] Pablo Pedregal. *Variational methods in nonlinear elasticity*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000.
[29] Mario Pitteri and Giovanni Zanzotto. *Continuum models for twinning and phase transitions in crystals*. Chapman and Hall, London, 1996.
[30] Y. C. Shu. Heterogeneous thin films of martensitic materials. *Arch. Rational Mech. Anal.*, 153:39–90, 2000.

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