Repellers for the Laguerre Iteration Function

Pavel Bělík and Vasileios Drakopoulos*

Abstract For polynomials some of whose zeros are complex, little is known about the overall convergence properties of Laguerre's function. This chapter provides an outline of this function viewed as a dynamic system which is often studied by many researchers and includes some of the latest research made by the authors. Moreover, the existence of its free critical points as applied to one-parameter families of quadratic and cubic polynomials is examined. With the help of computer-generated plots, we investigate the basins of attraction of the zeros in the special case wherein this function is constructed to converge to the nth roots of unity. Furthermore, we examine the effect and the impact of perturbing the symmetry of these roots on their basins of attraction.

Key words: attractor; basin of attaction; dynamic system; iterative method; Laguerre's function; repeller

1 Introduction

Laguerre iteration function is by far the most straightforward iterative method for finding roots of polynomials all of whose zeros are real and simple. If all the zeros are real but some are not simple, the method still converges but the convergence is of first order in the neighborhood of a multiple zero. For polynomials with real roots, the real line is divided into as many abutting intervals as there are distinct roots and

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from any initial point in such an interval the successive Laguerre iterates converge monotonically to the root therein; see [20] and the references therein.

In this work we are interested in Laguerre iterations on the complex plane. The previous property does not generally extend to the complex case as it stands, i.e., generally the complex plane is not covered by abutting regions such that from an initial value in a region successive iterates converge to the zero contained therein.

Some steps in the derivation of Laguerre's method for approximating the roots of a polynomial equation that are normally omitted from the few texts that discuss the method is described in [26]. A one-parameter family of iteration functions for finding roots is derived in [14]. Recent applications of some old work of Laguerre can be found in [13]. A generalization of Laguerre's method to higher-order methods that have the same desirable global convergence properties that Laguerre's method does for polynomials with real zeros is presented in [12]. The article [19] has been written to explain its properties in an elementary fashion. The article [11] supplements the theoretical background of the quasi-Laguerre iteration by including the proofs of the convergence properties. An attempt to bring Laguerre's method to the high level of robustness and effectiveness as the Cluster-Adapted Method package can be found in [5].

A substantiation of a new Laguerre's type iterative method for solving of systems of nonlinear polynomial equations with real coefficients is presented in [22]. The problems of its implementation, including relating to the structural choice of initial approximations, were also considered. A family of Laguerre methods of order three as well as two others of the same order is compared in [18]. The conjugacy maps and the effect of the extraneous roots on the basins of attraction is also discussed. Symmetry properties of the Laguerre iteration function are studied in [2] and the dynamics of the method is also clarified there. A one-parameter Laguerre's family of iterative methods for solving nonlinear equations is considered in [21]. In that article the authors compare convergence characteristics of Laguerre's family of iterative methods $L(x;\lambda)$ for various values of the real parameter λ using three different methodologies. An implementation of a modified Laguerre method for the simultaneous approximation of all roots of a polynomial is presented in [3]. Other applications can be found in [7].

As far as the dynamics of the Laguerre function in the complex plane is concerned, we demonstrate the non-existence of critical points which may be trapped by an iteration sequence associated with one-parameter families of quadratic and cubic polynomials in Section 5. Moreover, a brief examination of a one-parameter family of cubic polynomials is included. In Section 6 we examine, with the aid of computer-generated plots, the basins of attraction of this method constructed to converge to the nth roots of unity and, subsequently, the associated Julia sets. In the specific case of two roots, the roots' basins of attraction are separated by the right bisector of the line joining the two roots, which is the same scenario as the classical result of Cayley's for Newton's method [4]. In the case of three roots, the roots' basins of attraction are the connected regions separated by the lines bisecting the edges of the triangle with the roots as vertices. A similar result is also valid in the case of four roots. In both of these cases, Laguerre's method converges globally, including from the boundaries of the individual basins, except when the starting value is $z_0 = 0$. For $n \geq 5$, the dynamics of the Laguerre function is much more complicated and the boundary of the union of the roots' basins of attraction is fractal. Finally, in Section 7 we explore the effect the symmetry of the roots of these polynomials might be playing in the poor performance of Laguerre's method and observe that a small perturbation may result in a bifurcation leading to an overall excellent performance.

2 Laguerre's Iterative Method

Section 9.5 of [23] claims that Laguerre's method, used for finding zeros of a polynomial, gives strong convergence right from any starting value. According to Ralston and Rabinowitz [24], however, this is only true if all the roots of the polynomial are real. For example, Laguerre's method runs into difficulty for the polynomial $f(x) = x^n + 1$ with $n > 2$ if the initial guess is 0, because $f'(0) = f''(0) = 0$. The method can be extended to the complex plane as follows, but refer also to [20].

Let f be analytic in some region T, let $w \in T$ be a zero of f and let $f'(w) \neq 0$. Let ν be a real number, $\nu \neq 0, 1$. Then, there exists a neighborhood D of w such that

$$\left| \frac{\nu}{\nu - 1} \frac{f(z) f''(z)}{f'(z)^2} \right| < 1, \quad z \in D.$$

Consequently the square root

$$r(z) = \left[1 - \frac{\nu}{\nu - 1} \frac{f(z) f''(z)}{f'(z)^2} \right]^{1/2}$$

is analytic in D and can be defined by its principal value; furthermore

$$r(z) = 1 - \frac{\nu}{2(\nu - 1)} \frac{f(z) f''(z)}{f'(z)^2} + O((z - w)^2).$$

For $z \in D$ we define

$$L(z) = z - \frac{f(z)}{f'(z)} \frac{\nu}{1 + (\nu - 1) r(z)} \tag{1}$$

and assert the following.

Theorem 1 *For every $\nu \neq 0, 1$ the function L defined by* (1) *is an iteration function of order 3 for solving $f(z) = 0$.*

Proof See [15], p. 532. □

This means that, in the neighborhood of a simple root of f, iteration of L converges at least cubically.

The *Laguerre iteration function* $L(z)$ is defined as (see [15])

$$L(z) = z - \frac{\nu f(z)}{f'(z) + \left[(\nu-1)^2 f'(z)^2 - \nu(\nu-1) f(z) f''(z)\right]^{1/2}}, \tag{2}$$

where the argument of the root is to be chosen to differ by at most $\pi/2$ from the argument of $f'(z)$, or equivalently, to maximize the modulus of the denominator in (2). When both choices satisfy this criterion, a choice is typically made at the algorithmic level. The iteration function (2) can also be rewritten in an equivalent form as

$$L(z) = z - \frac{\nu f(z)/f'(z)}{1 + \left[(\nu-1)^2 - \nu(\nu-1) f(z) f''(z)/f'(z)^2\right]^{1/2}}. \tag{3}$$

We observe that $L(z) = \infty$ if and only if $f'(z) = f''(z) = 0$ and $f(z) \neq 0$. For $\nu = 3$ the iteration (3) becomes

$$L(z) = z - \frac{2 f(z)/f'(z)}{1 + \left[1 - 2 f(z) f''(z)/f'(z)^2\right]^{1/2}}. \tag{4}$$

From this formula we may derive more simple ones by introducing approximations. For example, if in (4) we introduce the approximation

$$\sqrt{1 - 2\,\frac{f(z) f''(z)}{f'(z)^2}} \approx 1 - \frac{f(z) f''(z)}{f'(z)^2},$$

we obtain the formula

$$H(z) = z - \frac{f(z)/f'(z)}{1 - f(z) f''(z)/\left[2 f'(z)^2\right]}$$

which is also of third order, but which does not require a square-root computation. This is the frequently rediscovered formula of *Halley*. Iterative approximation based on this formula is also sometimes called *Bailey's method* or *Lambert's method*. This formula belongs to a more general class of iterative functions called *König's functions* and is specifically $K_3(z)$. We refer the interested reader to [1] or [8].

If we write

$$\left[1 - \frac{f(z) f''(z)}{2 f'(z)^2}\right]^{-1} \approx 1 + \frac{f(z) f''(z)}{2 f'(z)^2}$$

in Halley's formula, we obtain the iteration

$$C(z) = z - \frac{f(z)}{f'(z)} \left[1 + \frac{f(z) f''(z)}{2 f'(z)^2}\right].$$

This is also a third-order iteration sometimes called *Chebyshev's formula*. For more details on this subject see [16]. This formula also belongs to a more general class of iterative functions called *Schröder's functions* and is specifically $S_3(z)$ (see for example [10]).

The Laguerre iteration function phenomenologically offers no advantages over Schröder's S_3, which is also somewhat easier to compute. From a practical point of view, however, the presence of the square root has the desirable effect that if f is a real polynomial the iteration automatically branches out into the complex plane if no real roots are found. Moreover, if f is a real polynomial of degree $n \geq 2$, the choice $\nu = n$ furnishes remarkable inclusion theorems for the real zeros, such as

Theorem 2 *Let f be a polynomial of degree n and let L be the Laguerre iteration function formed with $\nu = n$. Then for each complex number z there is a zero w of f such that $|w - z| \leq \sqrt{n}\,|L(z) - z|$.*

Proof See [17]. □

3 Laguerre Iteration Function Revisited

None of the derivations of (2) or (3) is easy to motivate. One typical derivation is as follows. If $f(z) = (z - \rho_1)(z - \rho_2) \cdots (z - \rho_n)$, then

$$F(z) = \frac{f'(z)}{f(z)} = \frac{1}{z - \rho_1} + \frac{1}{z - \rho_2} + \cdots + \frac{1}{z - \rho_n} = \frac{d}{dz} \ln |f(z)|$$

and

$$G(z) = F(z)^2 - \frac{f''(z)}{f(z)} = \frac{1}{(z - \rho_1)^2} + \frac{1}{(z - \rho_2)^2} + \cdots + \frac{1}{(z - \rho_n)^2} = -\frac{d^2}{dz^2} \ln |f(z)|.$$

Then, for z near a simple root ρ, the other roots are "far away" and so $f(z) \approx (z - \rho)(z - \xi)^{n-1}$ for some ξ. Thus

$$F(z) \approx \frac{1}{z - \rho} + \frac{n - 1}{z - \xi} \qquad G(z) \approx \frac{1}{(z - \rho)^2} + \frac{n - 1}{(z - \xi)^2}.$$

Assuming the approximations are exact yields a solution

$$z - \rho = \frac{n}{F(z) \pm \sqrt{(n - 1)\left[nG(z) - F(z)^2\right]}},$$

which, with the sign chosen to maximize the modulus of the denominator, leads immediately to (2). So, if

$$\Lambda(z) = \frac{F(z) + \delta(z)\sqrt{\nu - 1}}{\nu},$$

where $\delta(z) = \pm\sqrt{\nu G(z) - F(z)^2}$ with the sign of chosen to maximize $|\Lambda|$, then (2) becomes

$$L(z) = z - \frac{1}{\Lambda(z)}.$$

4 Dynamic Systems Approach

In what follows, we abbreviate as f^k the k-fold composition $f \circ f \circ \underset{k}{\cdots} \circ f$ and by *region* we mean a connected open set on the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. It appears that, for cases of practical interest, convergence of the sequence of iterates z_0, $f(z_0)$, $f^2(z_0)\ldots$ is assured for every choice of starting point z_0 in the complex plane, except when z_0 is a point of the Julia set $J(f)$. How "close" a starting point must be to the desired root depends on certain convergence conditions and how "fast" the method converges depends on the order of convergence of our iterative method.

It may happen, however, that when choosing a starting point z_0 in a certain domain, convergence takes place not to a zero of our polynomial, but to a *periodic orbit* or *cycle*, that is a set of $p \geq 2$ distinct points $\{a_1, \ldots,\ a_p\}$ such that

$$f(a_1) = a_2,\ \ldots,\ f(a_{p-1}) = a_p,\ f(a_p) = a_1,$$

so that, in fact, for each $k = 1, 2, \ldots, p$, $z = a_k$ is a solution of $f^p(z) = z$. If $p = 1$, z is called a *fixed point* of f. Hence, a point a is *periodic* if $f^p(a) = a$ for some $p > 0$; it is *repelling, indifferent* or *attracting* depending on whether $|(f^p)'(a)|$ is greater than, equal to, or less than one.

If a is an attracting fixed point of f, then

$$A(a) = \{z \in \mathbb{C} : \lim_{k\to\infty} f^k(z) = a\}$$

is the *basin of attraction* of a. If a basin of attraction is not connected, we often wish to consider the *immediate basin of attraction* $A^*(a)$ of a, namely the connected component of $A(a)$ which contains a itself. For our purposes we will need to consider the union of the basins of attraction of all fixed points of f; its boundary is the *Julia set* $J(f)$ of the function f.

It is well known that the dynamics of polynomials and rational maps is determined to a large extent by the fate of the orbits of critical values. *Critical values* of a function f are defined as those values $v \in \mathbb{C}$ for which $f(z) = v$ has a multiple root. The multiple root $z = c$ is called the *critical point* of f. This is equivalent to the condition $f'(c) = 0$. In some cases, such as in $E_\lambda(z) = \lambda e^z$, $\lambda \in \mathbb{R}$ that has no critical points, the role of the critical value is played by the *asymptotic value* 0, which is an omitted value for E_λ. In this article we intend to exclude the case in which some critical point will converge to an attracting cycle, should such a cycle exist.

Among the critical points of the Laguerre function L, determined by the condition $L'(z) = 0$, are the zeros z_i^* of f, which are also attracting fixed points of L. These points are obviously not free to converge to any other attracting cycles. In the next section we seek the existence of other roots of L', which we shall call the *free critical points*.

Following the notation of this section, the condition $L'(z) = 0$ implies that

$$\Lambda(z)^2 + \Lambda'(z) = 0,$$

which yields

$$\begin{aligned}3(\nu-1)(\nu-2)^2 f'(z)^2 f''(z)^2 - 4(\nu-1)^2(\nu-2) f'(z)^3 f'''(z)\\ - 4\nu(\nu-2)^2 f(z) f''(z)^3 + 6\nu(\nu-1)(\nu-2) f(z) f'(z) f''(z) f'''(z)\\ - \nu^2(\nu-1) f(z)^2 f'''(z)^2 = 0.\end{aligned} \tag{5}$$

We will use this equation in Section 5 below, where we study the behavior of Laguerre's iteration for some specific polynomials.

The following statement is often found in the literature but never rigorously proved because no such proof exists: *"The Laguerre iteration function remains invariant under every Möbius transformation."* We next present a counterexample to this statement and a weaker, but correct statement, both due to Ray [25].

Consider the polynomial p and a Möbius transformation T given by

$$p(z) = (z-1)(z-2)(z+3), \qquad T(z) = \frac{z}{2z-3}.$$

We then have $p(0) = 6$, $p'(0) = -7$, $p''(0) = 0$ and denoting the Laguerre iteration function (2) with $\nu = \deg p = 3$ for p by L_p we have

$$L_p(0) = \frac{6}{7}.$$

Since $T(1) = -1$, $T(2) = 2$, and $T(-3) = 1/3$, the transformed polynomial and the transformed starting value $z = 0$ are

$$g(z) = (z+1)(z-2)(z-1/3), \qquad T(0) = 0.$$

We then have $g(0) = 2/3$, $g'(0) = -5/3$, $g''(0) = -8/3$, and

$$L_g(T(0)) = L_g(0) = \frac{6}{19} \neq T\left(\frac{6}{7}\right) = -\frac{2}{3}.$$

Therefore the statement above is not correct.

The reason why the invariance is not correct in general is the particular choice of the argument in the square root of (2). Ray [25] shows that the invariance is true for affine maps $T(z) = az+b$ or, for example, if the choice of the sign of the square root is irrelevant, i.e., for those values of z for which $(n-1)^2 f'(z)^2 - n(n-1) f(z) f''(z) = 0$.

5 Parameter space

We now focus attention on the Laguerre iteration method associated with the quadratic family

$$p_c(z) = z^2 - c, \quad c \in \mathbb{C},$$

and with the particular one-parameter family of cubic polynomials,

$$p_\lambda(z) = z^3 + (\lambda - 1)z - \lambda = (z-1)(z^2 + z + \lambda), \quad \lambda \in \mathbb{C}$$

the zeros of which are $z_1^* = 1$, $z_2^* = (-1 + \sqrt{1-4\lambda})/2$ and $z_3^* = (-1 - \sqrt{1-4\lambda})/2$. Note the λ-dependence of z_2^* and z_3^*. The polynomials p_λ are exactly the monic cubics whose roots sum to zero and which have 1 as a root. Since any quadratic can be transformed into a p_c and any cubic can be transformed into a p_λ or into z^3 by an affine change of the variable and multiplication by a constant, thus analysing Laguerre's method for a general quadratic or cubic reduces essentially to analyzing it for the p_c's or p_λ's, respectively.

One main question is, are there any regions in the parameter spaces where attracting periodic cycles exist in addition to the (attracting) fixed points associated with the zeros of p_c or p_λ? To detect the existence of attracting cycles which could interfere with the Laguerre search for the roots z_i^* we examine the existence of the free critical points of the L function.

From (5) and setting $\nu = \deg p_c = 2$, the free critical points (should there be any) for the L function associated with p_c can be derived from

$$p_c(z)^2 p_c'''(z)^2 = 0, \tag{6}$$

which holds for every $z \in \mathbb{C}$. Also, it is straightforward to check by direct substitution of p_c into L that with $\nu = 2$ and $c \neq 0$ the iteration converges to one of the two roots in one step from any initial value in the complex plane. When $c = 0$, the same is true for any nonzero initial value.

From (5) the free critical points for the L function associated with p_λ can be derived from

$$\begin{aligned} 6(n-3)^2(n-4)z^6 - 6(n-3)^2(n+2)(\lambda-1)z^4 + 12n(n-3)(2n-5)\lambda z^3 \\ + 6n(n-1)(n-3)(\lambda-1)^2 z^2 - 12n(n-1)(n-3)\lambda(\lambda-1)z \\ - 2(n-1)^2(n-2)(\lambda-1)^3 - 3n^2(n-1)\lambda^2 = 0. \end{aligned}$$

Setting $n = \deg p_\lambda = 3$ in the above equation we deduce that for two values of λ, specifically for $\lambda = -2$ with multiplicity two and $\lambda = 1/4$, this also holds for every $z \in \mathbb{C}$, as in (6). We conclude that there aren't any free critical points, something that suggests that the dynamics of Laguerre's method for all the complex quadratic and cubic polynomials is unaffected by critical points.

A square grid corresponding to 200×200 pixels of a monitor represent a region in the complex plane. After each iteration, the Euclidean distances between the iterate z_k and the zeros z_i^* of $f_n(z)$ (or of $p_\lambda(z)$) were computed. If any of the distances were less than 0.0001, it was assumed that the sequence would converge to that particular root. If after 200 iterations no such convergence was observed, the routine would skip to the next grid point. The basin of attraction $A(z_i^*)$ for each root of unity z_i^* would be assigned a characteristic color. For $\lambda = -2$, blue regions constitute $A(1)$; green regions constitute $A(-2)$. For $\lambda = 1/4$, blue regions constitute $A(1)$; green regions constitute $A(-1/2)$. For $\lambda = i$, blue regions constitute $A(1)$; green regions constitute $A(-0.5 + 0.5\sqrt{1-4i})$; red regions constitute $A(-0.5 - 0.5\sqrt{1-4i})$. The common boundaries of these basins of attraction constitute the points in $\mathbb{C}$ where the algorithmic decision discussed after (2) has to be made, and convergence to one of the roots occurs depending on the choice.

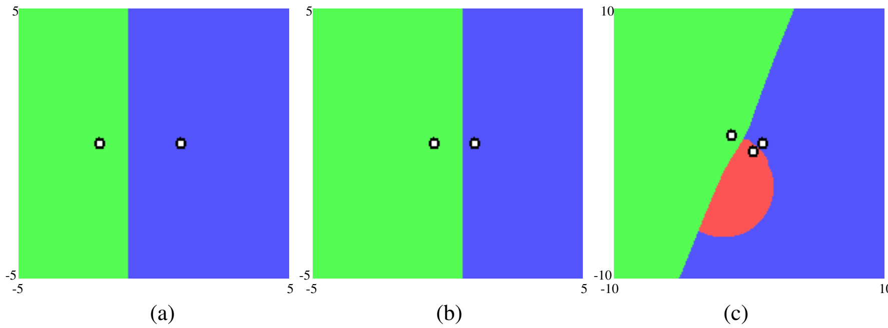

Fig. 1 Basins of attraction for the roots of $p_\lambda(z) = z^3 + (\lambda - 1)z - \lambda$ using Laguerre function. (a) $\lambda = -2$, (b) $\lambda = 1/4$, (c) $\lambda = i$. [9]

6 Roots of Unity Under Laguerre's Iteration with $\nu = n$

We now examine in more detail the behavior of the Laguerre iteration function when applied to find the nth roots of unity with $n \geq 2$. Additional details can be found in [25, 2, 6]. Substituting $f_n(z) = z^n - 1$ into the Laguerre iteration function (3) with $\nu = n$, we can simplify the resulting expression to get

$$L(z) = z\,\frac{\frac{1}{\sqrt{z^n}} + (n-1)}{\sqrt{z^n} + (n-1)}, \tag{7}$$

where $\sqrt{z}$ denotes the principal square root of $z \in \mathbb{C}$ and where also $L(0) = 1$ and $L(\infty) = 0$ for $n = 2$, and $L(0) = \infty$ and $L(\infty) = 0$ for $n \geq 3$.

We note that the roots of f_n are exactly the fixed points of L, and it is easy to check that the derivative of L vanishes at the roots, so they are attracting fixed points, and each has an open neighborhood contained in its immediate basin of attraction. When $n = 2$, which is a special case of the polynomial p_c discussed in Section 5, convergence occurs in one iteration for any starting point $z_0 \in \mathbb{C}$. If $\mathrm{Re}(z_0) > 0$, or if $\mathrm{Re}(z_0) = 0$ and $\mathrm{Im}(z_0) \geq 0$, then $L(z_0) = 1$; otherwise $L(z_0) = -1$. When $n \geq 3$, $z = 0$ is a repelling periodic point of period 2, because $L(0) = \infty$ and $L(\infty) = 0$. For any other starting point convergence to the roots of f_n occurs for $n = 3$ and $n = 4$. The basins of attractions to the roots for $n = 2$, 3, 4 are shown in Fig. 2 and as indicated in the figure, the boundaries of the roots' basins of attraction consist of the Voronoi diagram of the corresponding roots. Again, these boundaries are points at which an algorithmic choice of the sign of the square root in (2) would be made. With the iteration function (7) this choice is made by the machine's implementation of the principal square root.

The situation is much more interesting for $n \geq 5$. In this case, the $\{0, \infty\}$ two-cycle becomes attracting and the Lebesgue measure of the roots' basins of attraction approaches 0 as $n \to \infty$. More specifically, these basins are subsets of an annulus whose radii r_1 and r_2 satisfy $0 < r_1 < 1 < r_2 = 1/r_1 < (n-1)^{2/(n-4)}$, so both radii converge to 1 as $n \to \infty$. In Fig. 3 the case with $n = 16$ is shown with the 16 roots shown as dots along the unit circle, and two grey annuli that vaguely outline the boundary of the union of the basins of attraction of the roots: Laguerre iteration converges to the roots for any starting point between the two grey annuli, but it approaches the $\{0, \infty\}$ two-cycle for any starting point inside the smaller grey annulus or outside the larger grey annulus. We illustrate the intricacies of the basins of attraction to the roots in Fig. 4 in which the cases with $n = 5$, 6, 7, and 8 are shown. Note that all of these images are shown to display the entire union of the basins of attraction scaled to fit into the viewing window, so the scale is different for all of them.

The boundary of the union of the n basins of attraction to the roots of f_n, or the Julia set $J(f_n)$, is contained in the grey annuli shown in Fig. 3. It turns out that in general the union of the n basins is neither connected nor simply connected and the Julia set exhibits fractal and quasi-self-similar structure. These concepts are illustrated in Fig. 5 in which zooms into two parts of the boundary are shown for f_n with $n = 128$. We note that more intricate structure develops as n increases. (For an additional example with $n = 1024$ see [2].)

It is important to point out that if the Laguerre iteration function (2) is used instead of its mathematical equivalent (7) to iterate on a computer with floating-point arithmetic, the results will be different. The colors shown in Figs. 4 and 5 would remain unchanged, but where one currently sees white regions a mix of all colors would appear giving the appearance of Laguerre's method converging from

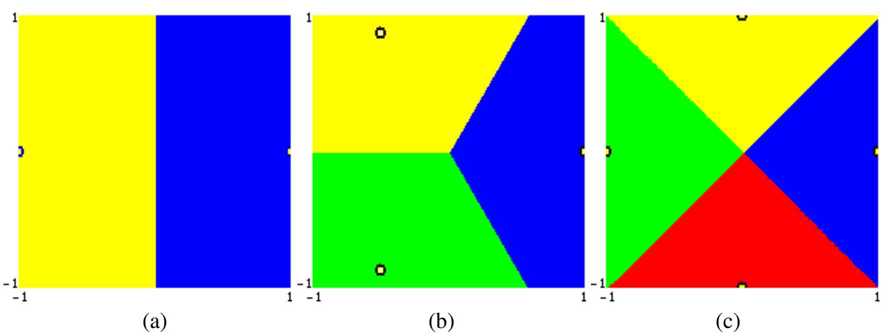

Fig. 2 Basins of attraction for the roots of $f_n(z) = z^n - 1$ in the complex region $[-1, 1] \times [-1, 1]$ using Laguerre function: (a) $n = 2$, (b) $n = 3$, (c) $n = 4$. [10]

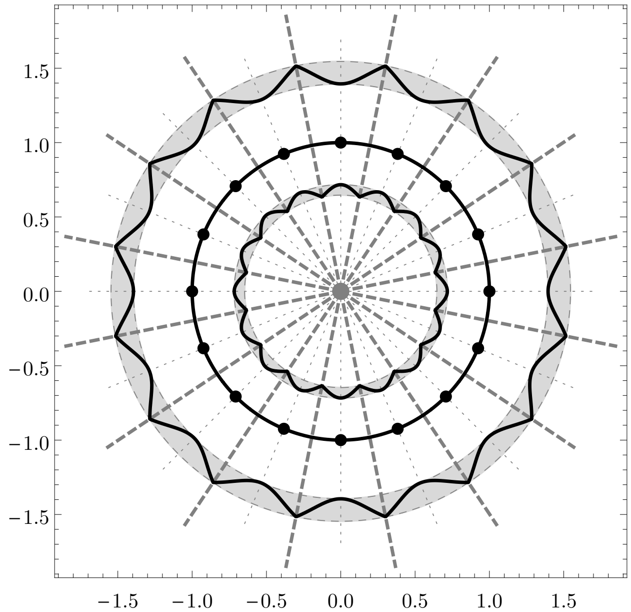

Fig. 3 Depiction of various sets and curves relevant in the analysis of the Laguerre iteration function for $f_n(z) = z^n - 1$ with $n = 16$. Convergence to roots occurs when starting between the grey annuli; convergence to the $\{0, \infty\}$ two-cycle occurs when starting inside the smaller grey annulus or outside the larger grey annulus; the boundary of the union of the basins of attraction is in the grey region. [2]

any starting point in $\mathbb{C}$. This is illustrated in Fig. 6. That is, in fact, what would take place in the computer and the reason is loss of significance in the computation of the expression $(n-1)^2 f_n'(z)^2 - n(n-1) f_n(z) f_n''(z)$ in (2) when $|z|$ is large. Note that both terms in the difference have leading terms $n^2(n-1)^2 z^{2n-2}$, and the actual difference should be equal to $n^2(n-1)^2 z^{n-2}$, several orders of magnitude smaller. We therefore see that, for large $|z|$, significant errors will occur in the computation of the square root in (2). When convergence to the two-cycle $\{0, \infty\}$ should theoretically take place, eventually the large magnitude of the alternate iterates will result in erroneous values from which convergence to roots occurs. We note that this loss of significance is not unique to $f_n(z) = z^n - 1$. For a general polynomial $p(z)$ of degree n the difference $(n-1)^2 p'(z)^2 - n(n-1) p(z) p''(z)$ will have a leading term of order z^{2n-4} or smaller, while the leading terms of both $(n-1)^2 p'(z)^2$ and $n(n-1) p(z) p''(z)$ are equal and of order z^{2n-2}. Perhaps this observation explains the popular notion that Laguerre's method seems to converge to a root from almost any initial guess in $\mathbb{C}$.

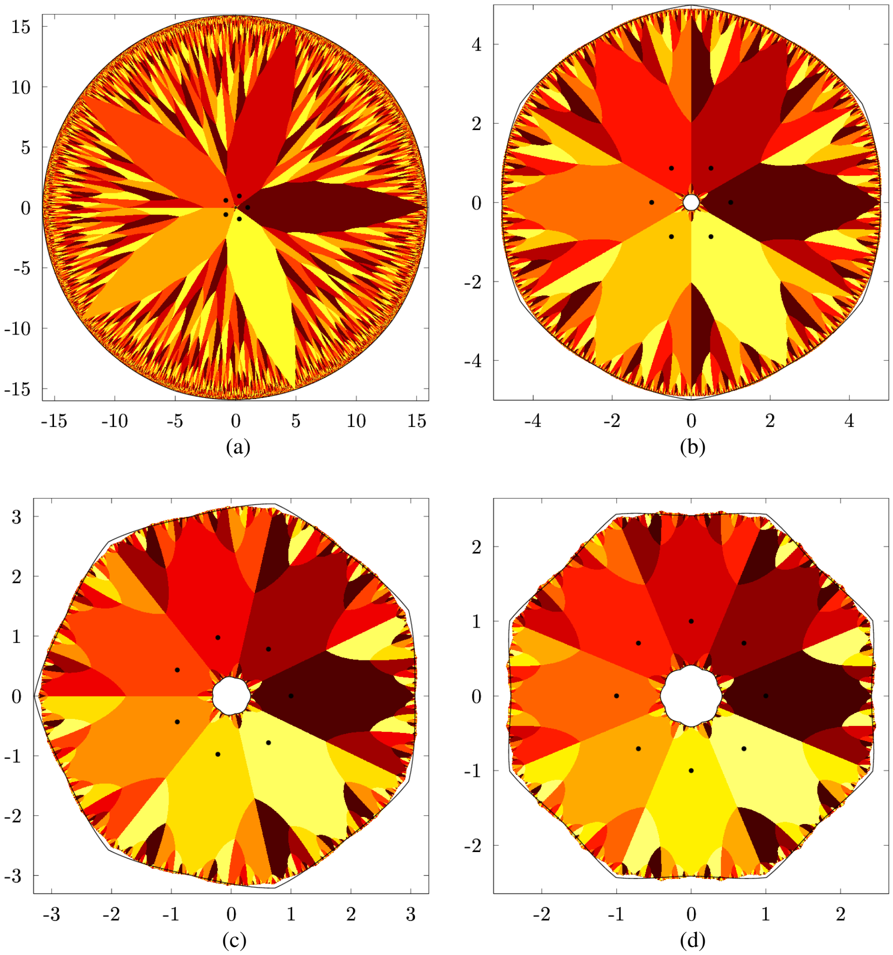

Fig. 4 Numerically computed basins of attraction of Laguerre's method applied to the polynomials $f_n(z) = z^n - 1$ with $n = 5, 6, 7,$ and 8 ((a)-(d), respectively). Each color corresponds to a basin of attraction of a root in the basin. The dots represent the roots of f_n and the two black curves in each image are the curves shown in Fig. 3 in the grey regions. [2]

7 Symmetry Effects

In this section we briefly preview work in progress on the potential effects of symmetry on the behavior of the Laguerre iteration (2) with $\nu = \deg f = n$. The polynomials f_n studied in the previous section have roots distributed uniformly and symmetrically on the unit circle. In Figs. 7 and 8 we show the roots' basins of attraction for the polynomials $q_1(z) = (z^5 - 1)(z^5 - 32)$ and $q_2(z) = (z^5 - 1)(z^5 + 32)$. These polynomials have their roots symmetrically distributed along the circles of radii 1 and 2, with q_1 having the real roots 1 and 2 and q_2 having the real roots 1 and -2.

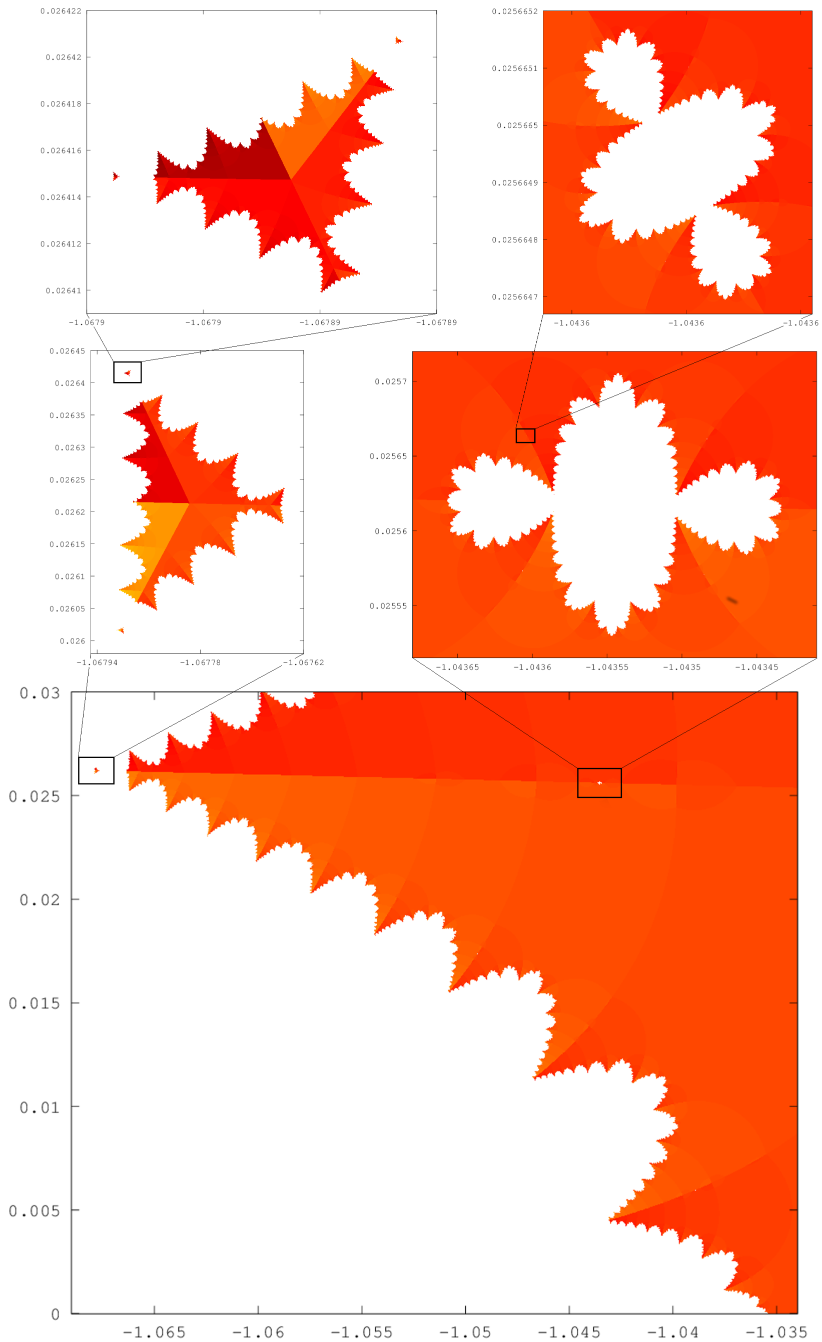

Fig. 5 Several consecutive zooms into two parts of the boundary of the union of the basins of attractions to the roots of $f_n(z) = z^n - 1$ for $n = 128$. [2]

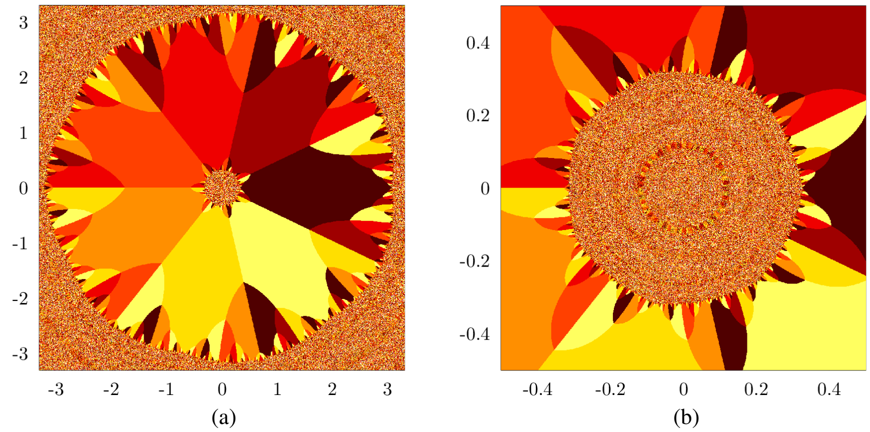

Fig. 6 Computed basins of attraction for $f_7(z) = z^7 - 1$ using the general formulation of the method (2): (a) the equivalent of Fig. 4(c); (b) zoom into the center part of (a). [2]

Theoretical considerations and the numerical results show that the Laguerre iteration has similar dynamics as for the case with $f_n(z) = z^n - 1$ in the previous section: in both cases the union of the basins of attraction to the roots is a subset of an annulus centered at the origin and $\{0, \infty\}$ is an attracting two-cycle.

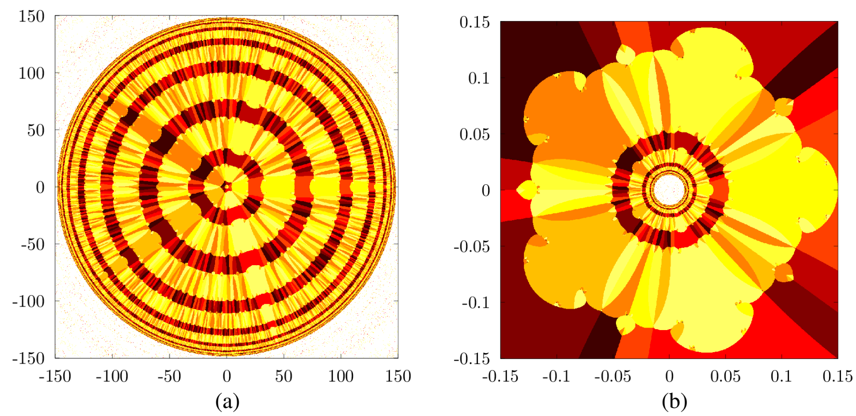

Fig. 7 The roots' basin of attraction for $q_1(z) = (z^5 - 1)(z^5 - 32)$: (a) the whole basin; (b) zoom into the center region.

These observations then motivate the question of how symmetry of the polynomials might be affecting the performance of the Laguerre iteration. A first exploration of this question concerns the idea of perturbing one of the roots of f_n to break the symmetry of the roots. To this end, consider the polynomial $p_r(z) = (z - r)(z^4 + z^3 + z^2 + z + 1)$ in which r is viewed as a perturbation of

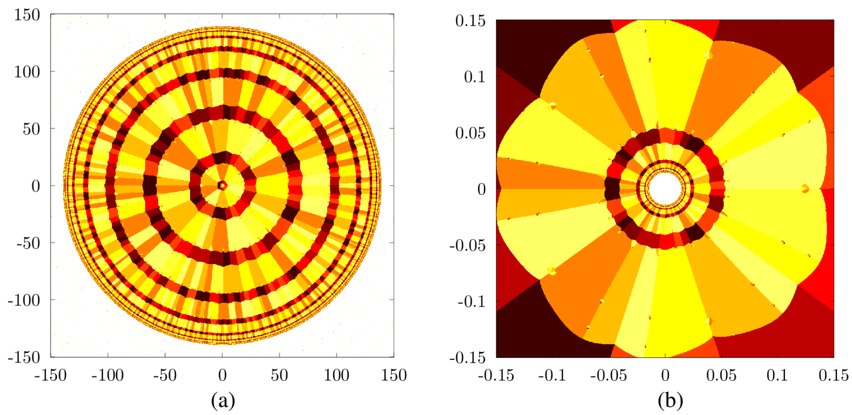

Fig. 8 The roots' basin of attraction for $q_2(z) = (z^5 - 1)(z^5 + 32)$: (a) the whole basin; (b) zoom into the center region.

the real root $z^* = 1$ of $f_5(z) = z^5 - 1$. In Figs. 9 and 10 we present the results of small perturbations with $r \in \mathbb{R}$ and the consequent changes in the roots' basins of attraction. Note how small perturbations result in significant changes in the basins. In Fig. 9 we show the original basin with $r = 1$ and two small perturbations: one with $r = 1.00060$ and one with $r = 1.00064$. Notice how sensitive the results are to these two perturbations. In the middle image the union of the basins of attraction has slowly grown and slightly changed its shape. When changing r from 1.00060 to 1.00064, there is a sudden change and as far as we could computationally check, the basins visibly cover the complex plane.

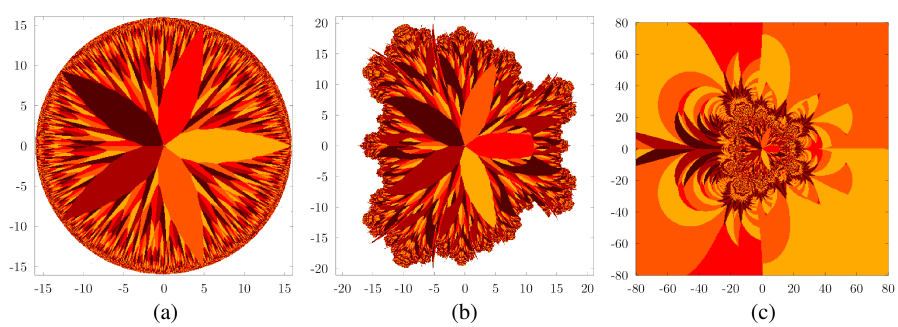

Fig. 9 Basins of attraction to the roots of p_r: (a) $r = 1$, (b) $r = 1.00060$, (c) $r = 1.00064$.

In Fig. 10 we show the effects of changing r to the values 0.99850 in (a), 0.99825 in (b), and 0.99822 in (c). A similar situation occurs here: first the union of the basins starts growing in a slow, controlled way, but the small change in r from 0.99825 to 0.99822 results in a sudden change again extending to much of the rest of the complex plane. Decreasing r further (not shown here) appears to extend the union of the basins of attraction to all but a possibly measure-zero subset of $\mathbb{C}$.

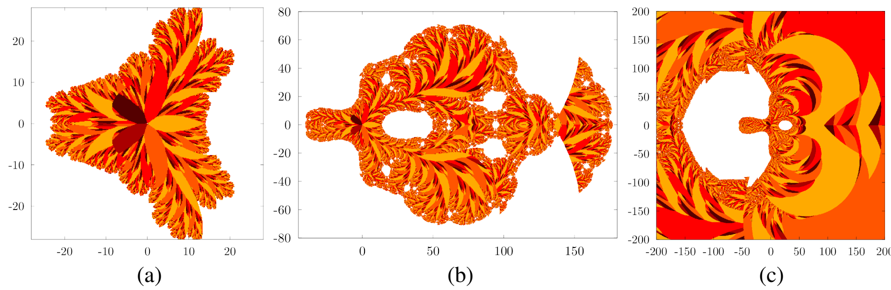

Fig. 10 Basins of attraction to the roots of p_r: (a) $r = 0.99850$, (b) $r = 0.99825$, (c) $r = 0.99822$.

We also note that if $r \neq 1$, then there is no longer the two-cycle $\{0, \infty\}$. Instead, new attracting cycles of even-length orbits seem to appear and the white regions in Figs. 9 and 10 correspond to the points in $\mathbb{C}$ converging to these cycles. The various bifurcations appearing in this problem are currently under investigation by the authors.

8 Conclusions

In practical problems there is often enough *a priori* knowledge of the desired roots of the equation to ensure that convergence of the iterations is not a problem. When *a priori* knowledge is poor, it is advisable to use a method that converges independently of the starting values, but perhaps slowly, and then switch to a more rapidly converging method.

In this work we reviewed some properties of the Laguerre iteration function (2) focusing mostly on the case with $\nu = \deg f = n$, where f is the polynomial whose roots are sought. We conclude that the iteration produces excellent results when applied to polynomials of degrees 2 and 3 as shown in Section 5 where the polynomials $p_c(z) = z^2 - c$ and $p_\lambda(z) = z^3 + (\lambda - 1)z - \lambda$ are studied.

From studying the very special case of the nth roots of unity we can conclude that there are polynomials for which convergence to the roots, at least when exact arithmetic is used, will not take place due to the existence of attracting cycles. This is the case for the polynomials $f_n(z) = z^n - 1$ with $n \geq 5$ for which $\{0, \infty\}$ is an attracting two-cycle, and many other polynomials with similar behavior, not discussed here, exist. For f_n with $n = 4$ the method converges globally except when starting from $z = 0$, but for $n \geq 5$ the measure of the union of the roots' basins of attraction tends to 0 as $n \to \infty$. However, this behavior may not be observed when floating-point arithmetic is used and even for $n \geq 5$ the method may converge globally as shown in Section 6. Additionally, when the symmetry of the roots of unity is slightly perturbed, convergence again seems to take place from much or all of the complex plane as shown in Section 7. The perturbations discussed here lead to the appearance of new attracting cycles and eventual bifurcations that appear to result in near-global convergence to the roots.

Unlike with rational iteration maps, under Laguerre's iteration the boundaries of the individual roots' basins of attraction do not correspond to the Julia sets of the polynomials. Instead, many of the boundaries shared by the basins are points from which convergence to one of the roots occurs and the boundary appears naturally due to the algorithmic choice the iteration presents when both choices of the square root in (2) result in denominators with the same modulus. As shown in Section 6, the Julia sets are only the fractal boundaries of the flower-like regions presented in Figs. 4–10, i.e., the unions of the roots' basins of attraction.

We conclude that even though for many polynomials of practical interest Laguerre's iteration has excellent results, there are some polynomials for which it may not. However, there are still many gaps in our understanding of the dynamics of the method and there is possibly a lot of potential that can still be unlocked.

Acknowledgments

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References

1. Argyropoulos, N., Böhm, A., Drakopoulos, V.: Julia and Mandelbrot-like sets for higher order König rational iteration functions. In: Novak, M.M., Dewey, T.G. (eds.) Fractal Frontiers, pp. 169–178. World Scientific, Singapore (1997)
2. Bělík, P., Kang, H., Walsh, A., Winegar, E.: On the dynamics of Laguerre's iteration method for finding the nth roots of unity. Int. J. Comput. Math. **2014**, Article ID 321585, 16 pages (2014)
3. Cameron, T.R.: An effective implementation of a modified Laguerre method for the roots of a polynomial. Numer. Algor. **82**, 1065–1084 (2019).
4. Cayley, A.: Application of the Newton–Fourier method to an imaginary root of an equation. Quart. J. Pure Appl. Math. **XVI**, 179–185 (1879)
5. Chen, T.C.: Improving Laguerre's method to cope with symmetry pitfalls in polynomial root-finding. In: Bainov D. (ed.) Proceedings of the Eighth International Colloquium on Differential Equations, pp. 105–110. VSP, Utrecht, the Netherlands (1997)
6. Curry, J.H., Fiedler, S.L.: On the dynamics of Laguerre's iteration: $Z^n - 1$. Physica D **30**, 124–134 (1988)
7. Datephanyawat, J., Torteeka, P., Anutarawiramkul, R., Plumanut, R., Palee, K., Prasit, P.: An improving capability of angles-only initial orbit determination via Laguerre root finding technique, Proc. SPIE 11755, Sensors and Systems for Space Applications XIV, 117550A (12 April 2021);

8. Drakopoulos, V.: How is the dynamics of König iteration functions affected by their additional fixed points? Fractals **7**, 327–334 (1999)
9. Drakopulos, V.: Are there any Julia sets for the Laguerre iteration function? Comput. Math. Appl. **46**, 1201-1210 (2003)
10. Drakopoulos, V., Argyropoulos, N., Böhm, A.: Generalized computation of Schröder iteration functions to motivate families of Julia and Mandelbrot-like sets. SIAM J. Num. Anal. **36**, 417–435 (1999)
11. Du, Q., Jin, M., Li T. Y., Zeng, Z.: The Quasi-Laguerre Iteration. Math. Comput. **66**, no. 217 345–61 (1997)
12. Foster, L.V.: Generalizations of Laguerre's method: Higher order methods. SIAM J. Num. Anal. **18**, 1004–1018 (1981)
13. Grosswald, E.: Recent applications of some old work of Laguerre. Am. Math. Mon. **86**, 648–658 (1979)
14. Hansen, E., Patrick, M.: A family of root finding methods. Numer. Math. 27, 257–269 (1976).
15. Henrici, P.: Applied and Computational Complex Analysis. John Wiley & Sons, New York (1974)
16. Hildebrand, F.B.: Introduction to Numerical Analysis, 2nd edn. McGraw-Hill, New York (1974)
17. Kahan, W.: Laguerre's method and a circle which contains at least one zero of a polynomial. SIAM J. Num. Anal. **4**, 474–482 (1967)
18. Neta, B., Chun, C.B.: On a family of Laguerre methods to find multiple roots of nonlinear equations, Appl. Math. Comput. **219**, 10987–11004 (2013)
19. Orchard, H.J.: The Laguerre method for finding the zeros of polynomials. IEEE T. Circuits Syst. **36**, 1377–1381 (1989)
20. Parlett, B.: Laguerre's method applied to the matrix eigenvalue problem. Math. Comput. **18**, 464–485 (1964)
21. Petković, L.D., Petković, M.S., Neta, B.: On optimal parameter of Laguerre's family of zero-finding methods. Int. J. Comput. Math. **96**, 692–707 (2019)
22. Poliakov, O., Pashkov, Y., Kolesova, M., Chepenyuk, O., Kalinin, M., Kramar, V.: New Laguerre's type method for solving of a polynomial equations systems, World Acad. Sci., Eng. Technol. **69**, 888–893 (2012)
23. Press, W.H., Flannery, B.P., Teukolsky, S.A., Vetterling, W.T.: Numerical Recipes: The Art of Scientific Computing. Cambridge University Press, New York (1986)
24. Ralston, A., Rabinowitz, P.: A First Course in Numerical Analysis, 2nd edn. McGraw-Hill, Singapore (1978)
25. Ray, T.L.: Laguerre's method for finding complex roots [Ph.D. thesis]. Stevens Institute of Technology, Hoboken, NJ, USA (1966)
26. Redish, K.A.: On Laguerre's method. Int. J. Math. Educ. Sci. Technol. **5**, 91–102 (1974)